Rossby Wave Speed Equation

A very useful equation for the speed of waves can be derived from the large-scale vorticity equation. First, assume horizontal flow and no horizontal divergence,

$$\frac{d_p(\zeta + f)}{dt} = 0.$$

Expand the total derivative,

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + v\beta = 0, \qquad (1)$$

where

$$\beta = \frac{\partial f}{\partial y} = 2\Omega \cos \phi \frac{\partial \phi}{\partial y} = \frac{2\Omega}{a} \cos \phi.$$

Assume the wind does not change in the north-south, or y, direction,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \text{ and } \frac{\partial \zeta}{\partial y} = 0.$$

Equation (1) becomes

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \beta v = 0.$$

Or

$$\frac{\partial^2 v}{\partial t \partial x} + u\frac{\partial^2 v}{\partial x^2} + \beta v = 0. \qquad (2)$$

Next, use the **perturbation method**.

1. Express dependent variables as the sum of a mean and a perturbation:

$$u(x,t) = U + u'(x,t)$$
$$v(x,t) = v'(x,t) \qquad (3)$$

2. Substitute equation (3) into the governing equation, (2),

$$\frac{\partial^2 v'}{\partial t \partial x} + (U + u')\frac{\partial^2 v'}{\partial x^2} + \beta v' = 0.$$

3. Linearize - neglect terms with products of perturbation (small) quantities.

$$\frac{\partial^2 v'}{\partial t \partial x} + U\frac{\partial^2 v'}{\partial x^2} + \beta v' = 0. \qquad (4)$$

Equation (4) is a single equation in one unknown, v' . Assume a solution,

$$v'(x,t) = A \sin\left[\frac{2\pi}{L}(x - ct)\right], \qquad (5)$$

a wave moving east with speed, c.

Next, substitute equation (5), (the assumed solution) into equation (4), using,

$$\frac{\partial v'}{\partial x} = \frac{2A\pi}{L}\cos\left[\frac{2\pi}{L}(x - ct)\right]$$

$$\frac{\partial^2 v'}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v'}{\partial x} = -\frac{4A\pi^2}{L^2}\sin\left[\frac{2\pi}{L}(x - ct)\right]$$

$$\frac{\partial}{\partial t}\frac{\partial v'}{\partial x} = \frac{4A\pi^2 c}{L^2}\sin\left[\frac{2\pi}{L}(x - ct)\right]$$

The result is

$$\frac{4A\pi^2 c}{L^2}\sin\left[\frac{2\pi}{L}(x - ct)\right] - U\frac{4A\pi^2}{L^2}\sin\left[\frac{2\pi}{L}(x - ct)\right] +$$

$$\beta A \sin\left[\frac{2\pi}{L}(x - ct)\right] = 0$$

Or,

$$\frac{4\pi^2 c}{L^2} - U\frac{4\pi^2}{L^2} + \beta = 0.$$

Or,

$$\frac{4\pi^2}{L^2}(c - U) + \beta = 0.$$

Or,

$$\boxed{c = U - \frac{\beta L^2}{4\pi^2}}\ . \qquad (6)$$

Equation (5) is a solution to equation (4), provided equation (6) is satisfied. It requires that all waves move slower than the mean wind, U, and that wave speed depends on wave length. Shorter waves move east more quickly than longer waves.