Ertel's Potential Vorticity

Because parcels rarely move along isobaric surfaces, the least accurate part of the last section is the use of pressure as the vertical co-ordinate. Parcels tend to move along isentropic surfaces because air flow is often adiabatic. Thus, potential temperature should be used as the vertical co-ordinate.

Start with a statement that parcels conserve their mass, M ,

$$\frac{dM}{dt} = 0.$$

Multiply the mass by one, volume divided by volume,

$$\frac{d}{dt}\left(\frac{MV}{V}\right) = 0.$$

Since $(M/V) = \rho$ and , $V = \Delta x\,\Delta y\,\Delta z$,

$$\frac{d}{dt}\left(\rho\,\Delta x\,\Delta y\,\Delta z\right) = 0.$$

Assume the atmosphere is hydrostatic, $\rho\,\Delta z = -\,\Delta p/g$,

$$\frac{d}{dt}\left(-\,\Delta x\,\Delta y\,\Delta p/g\right) = 0.$$

Multiply by $\Delta\,\theta\;/\;\Delta\,\theta$,

$$\frac{d}{dt}\left[\Delta x\,\Delta y\,\Delta\theta\left(\frac{-\,\Delta p}{g\,\Delta\theta}\right)\right] = 0.$$

Next, expand the total derivative using the distributive law.

$$\left[\Delta y\,\Delta\theta\left(\frac{-\,\Delta p}{g\,\Delta\theta}\right)\right]\frac{d}{dt}\left(\Delta x\right) + \left[\Delta x\,\Delta\theta\left(\frac{-\,\Delta p}{g\,\Delta\theta}\right)\right]\frac{d}{dt}\left(\Delta y\right) +$$

$$\left[\Delta x\,\Delta y\left(\frac{-\,\Delta p}{g\,\Delta\theta}\right)\right]\frac{d}{dt}\left(\Delta\theta\right) + \left[\Delta x\,\Delta y\,\Delta\theta\,\right]\frac{d}{dt}\left(\frac{-\,\Delta p}{g\,\Delta\theta}\right) = 0$$

Since $\frac{d}{dt}\left(\Delta x\right) = \Delta u$,

$$\left[\Delta y\ \Delta\theta\left(\frac{-\ \Delta p}{g\ \Delta\theta}\right)\right]\Delta u\ +\ \left[\Delta x\ \Delta\theta\left(\frac{-\ \Delta p}{g\ \Delta\theta}\right)\right]\Delta v\ +$$

$$\left[\Delta x\ \Delta y\left(\frac{-\ \Delta p}{g\ \Delta\theta}\right)\right]\Delta\frac{d\theta}{dt}\ +\ \left[\Delta x\ \Delta y\ \Delta\theta\ \right]\frac{d}{dt}\left(\frac{-\ \Delta p}{g\ \Delta\theta}\right)\ =\ 0$$

Since parcels conserve potential temperature, $\dfrac{d\theta}{dt}\ =\ 0$,

$$\left[\Delta y\ \Delta\theta\left(\frac{-\ \Delta p}{g\ \Delta\theta}\right)\right]\Delta u\ +\ \left[\Delta x\ \Delta\theta\left(\frac{-\ \Delta p}{g\ \Delta\theta}\right)\right]\Delta v\ +\ \left[\Delta x\ \Delta y\ \Delta\theta\ \right]\frac{d}{dt}\left(\frac{-\ \Delta p}{g\ \Delta\theta}\right)\ =\ 0.$$

Divide by $\Delta x\ \Delta y\ \Delta\theta$,

$$\left(\frac{-\ \Delta p}{g\ \Delta\theta}\right)\frac{\Delta u}{\Delta x}\ +\ \left(\frac{-\ \Delta p}{g\ \Delta\theta}\right)\frac{\Delta v}{\Delta y}\ +\ \frac{d}{dt}\left(\frac{-\ \Delta p}{g\ \Delta\theta}\right)\ =\ 0.$$

In the limit, as $\Delta\rightarrow 0$,

$$\left(\frac{-\ \partial\ p}{g\ \partial\ \theta}\right)\frac{\partial\ u}{\partial\ x}\ +\ \left(\frac{-\ \partial\ p}{g\ \partial\ \theta}\right)\frac{\partial\ v}{\partial\ y}\ +\ \frac{d}{dt}\left(\frac{-\ \partial\ p}{g\ \partial\ \theta}\right)\ =\ 0.$$

Or,

$$\left(\frac{-\ \partial\ p}{g\ \partial\ \theta}\right)\left(\frac{\partial\ u}{\partial\ x}\ +\ \frac{\partial\ v}{\partial\ y}\right)\ +\ \frac{d}{dt}\left(\frac{-\ \partial\ p}{g\ \partial\ \theta}\right)\ =\ 0,$$

or,

$$\left(\frac{\partial\ u}{\partial\ x}\ +\ \frac{\partial\ v}{\partial\ y}\right)\ =\ -\left(\frac{-\ \partial\ p}{g\ \partial\ \theta}\right)^{-1}\frac{d}{dt}\left(\frac{-\ \partial\ p}{g\ \partial\ \theta}\right). \qquad (1)$$

Equation (1) is the **isentropic equation of continuity for hydrostatic, adiabatic flow**.

Substitute for the horizontal divergence in the large-scale vorticity equation,

$$\frac{d(\zeta+f)}{dt}\ =\ +\ (\zeta+f)\left(\frac{-\ \partial\ p}{g\ \partial\ \theta}\right)^{-1}\frac{d}{dt}\left(\frac{-\ \partial\ p}{g\ \partial\ \theta}\right).$$

Divide by the absolute vorticity,

$$(\zeta+f)^{-1}\ \frac{d(\zeta+f)}{dt}\ =\ +\left(\frac{-\ \partial\ p}{g\ \partial\ \theta}\right)^{-1}\frac{d}{dt}\left(\frac{-\ \partial\ p}{g\ \partial\ \theta}\right).$$

Use the natural logarithm,

$$\frac{d}{dt}\ \ln(\zeta + f) \ = \ \frac{d}{dt}\ \ln\left(\frac{-\,\partial\, p}{g\,\partial\,\theta}\right).$$

As before,

$$\frac{d}{dt}\ \ln(\zeta + f) \ - \ \frac{d}{dt}\ \ln\left(\frac{-\,\partial\, p}{g\,\partial\,\theta}\right) \ = \ 0\,.$$

Or,

$$\frac{d}{dt}\ \ln\left[(\zeta + f)\left(-\,g\frac{\partial\,\theta}{\partial\, p}\right)\right] \ = \ 0\,.$$

Finally,

$$\boxed{\frac{d}{dt}\left[(\zeta + f)\left(-\,g\frac{\partial\,\theta}{\partial\, p}\right)\right] \ = \ 0} \qquad (2)$$

Equation (2) shows that **Ertel's potential vorticity**, P, is conserved,

$$P \ = \ (\zeta + f)\left(-\,g\frac{\partial\,\theta}{\partial\, p}\right).$$

This result is similar to that found in the last section, except that now a change of potential temperature, between the top and bottom of the layer, is explicitly shown. $\partial\,\theta\,/\partial\, p$ is a measure of the static stability, and is always negative. Accordingly, decreasing the stability (i.e., if $\partial\,\theta\,/\partial\, p$ becomes less negative), will increase the absolute vorticity.