Large-scale Vorticity Equation

The vorticity equation is an alternate form of the equation of motion that has many advantages. Begin with the equation of motion with pressure as the vertical co-ordinate,

$$\frac{du}{dt} = -g\frac{\partial z}{\partial x} + fv \quad \text{and} \quad \frac{dv}{dt} = -g\frac{\partial z}{\partial y} - fu.$$

Expand the total derivative,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial p} = -g\frac{\partial z}{\partial x} + fv, \qquad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \omega\frac{\partial v}{\partial p} = -g\frac{\partial z}{\partial y} - fu. \qquad (2)$$

Next, take the partial of equation (2) with respect to x and subtract the partial of equation (1) with respect to y.

The local derivative term becomes:

$$\frac{\partial}{\partial x}\frac{\partial v}{\partial t} - \frac{\partial}{\partial y}\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial \zeta}{\partial t}.$$

The horizontal advection terms become:

$$\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) =$$

$$u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + v\frac{\partial^2 v}{\partial y \partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} -$$

$$u\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - v\frac{\partial^2 u}{\partial y^2} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} =$$

$$u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) =$$

$$u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\zeta.$$

The vertical advection terms become:

$$\frac{\partial}{\partial x}\left(\omega\frac{\partial v}{\partial p}\right) - \frac{\partial}{\partial y}\left(\omega\frac{\partial u}{\partial p}\right) =$$

$$\omega\frac{\partial^2 v}{\partial x \partial p} + \frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} - \omega\frac{\partial^2 u}{\partial y \partial p} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p} =$$

$$\omega\frac{\partial}{\partial p}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p} =$$

$$\omega\frac{\partial \zeta}{\partial p} + \frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}.$$

The pressure gradient force terms are:

$$\frac{\partial}{\partial x}\left(-g\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(-g\frac{\partial z}{\partial x}\right) =$$

$$-g\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\right] = 0.$$

The Coriolis force terms become:

$$\frac{\partial}{\partial x}\left(-fu\right) - \frac{\partial}{\partial y}\left(fv\right) = -f\frac{\partial u}{\partial x} - u\frac{\partial f}{\partial x} - f\frac{\partial v}{\partial y} - v\frac{\partial f}{\partial y} =$$

$$-f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - u\frac{\partial f}{\partial x} - v\frac{\partial f}{\partial y}.$$

Putting all the terms together,

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\zeta + \omega\frac{\partial \zeta}{\partial p} +$$

$$\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - u\frac{\partial f}{\partial x} - v\frac{\partial f}{\partial y}.$$

Rearranging terms,

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \omega\frac{\partial \zeta}{\partial p} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} =$$

$$-\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(\zeta + f) - \frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} + \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}.$$

Or,

$$\frac{d\zeta}{dt} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(\zeta + f) - \frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} + \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}.$$

Since

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + \omega\frac{\partial f}{\partial p} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y},$$

$$\frac{d}{dt}(\zeta + f) = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(\zeta + f) - \frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} + \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}.$$

When the magnitude of each term is computed for large-scale circulations, the last two terms on the RHS, the product of relative vorticity and divergence, and the vertical advection of relative vorticity are found to be smaller than the other terms. Ignoring those terms yields the **scaled vorticity equation**,

$$\boxed{\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(\zeta + f) = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)f}\ .$$