Conservation of Vorticity

Using pressure as the vertical co-ordinate, the large-scale vorticity equation is

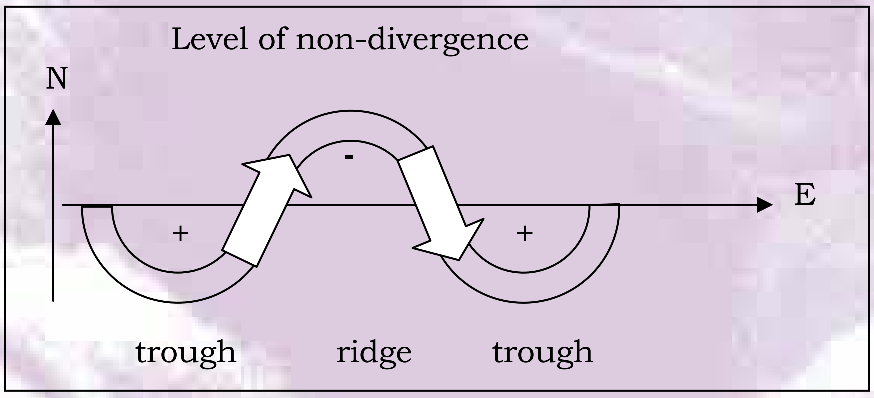
$$\frac{d_p(\zeta+f)}{dt} = -f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right). \qquad (1)$$

Absolute Vorticity

$$\boxed{\frac{d_p(\zeta+f)}{dt} = 0}$$

Applying equation (1) at the level of non-divergence (where the horizontal divergence is zero in the middle of the troposphere) leads to a statement that absolute vorticity is conserved.

The trajectory of a mid-troposphere, mid-latitude air parcel should have a strong



component towards the east from the prevailing winds. As the parcel conserves absolute vorticity, parcel relative vorticity will increase while moving towards the south (as the Earth's vorticity decreases), and will decrease while moving towards the north (as the Earth's vorticity increases). As parcel vorticity changes the parcel trajectory, a north-south oscillation results. This is one of the oldest and simplest explanations why waves occur in the mid-latitudes.

Potential Vorticity

Since horizontal divergence is usually zero only in the middle of the troposphere, it should be retained. Return to equation (1) and substitute for the horizontal divergence from the equation of continuity. Then, multiple both sides by dp

$$\frac{d(\zeta+f)}{dt}\,dp = (\zeta+f)\,\frac{\partial \omega}{\partial p}\,dp.$$

Assume ω varies more with p than with any other independent variable,

$$\frac{d(\zeta+f)}{dt}\,dp = (\zeta+f)\,d\omega.$$

Since

$$d\omega = \omega_2 - \omega_1 = \frac{dp_2}{dt} - \frac{dp_1}{dt} = \frac{d}{dt}\left(p_2 - p_1\right) = \frac{d}{dt}\,dp,$$

$$\frac{d(\zeta+f)}{dt}\,dp = (\zeta+f)\,\frac{d}{dt}\,dp.$$

Next, divide both sides by $(\zeta + f)\ dp$,

$$\frac{1}{(\zeta+f)}\,\frac{d\ (\zeta+f)}{dt} = \frac{1}{dp}\,\frac{d}{dt}\,dp.$$

And,

$$\frac{d\ \ln(\zeta+f)}{dt} = \frac{d\ \ln dp}{dt}, \qquad \text{or,} \qquad \frac{d\ \ln(\zeta+f)}{dt} - \frac{d\ \ln dp}{dt} = 0.$$

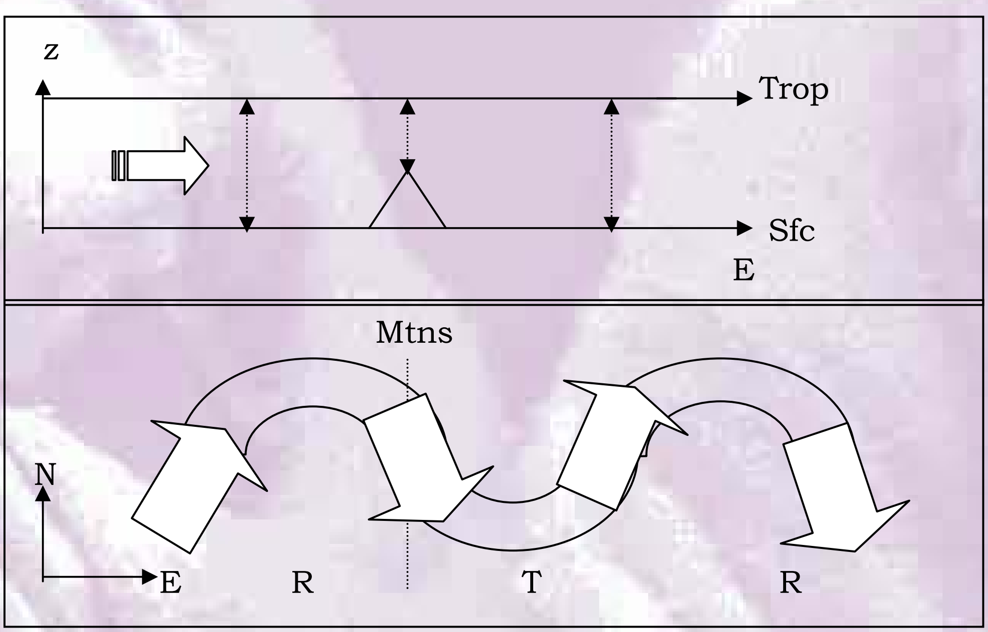
And,

$$\frac{d}{dt}\left(\ln\frac{(\zeta+f)}{dp}\right) = 0, \quad \text{or,} \quad \left(\frac{(\zeta+f)}{dp}\right)^{-1}\frac{d}{dt}\left(\frac{(\zeta+f)}{dp}\right) = 0,$$

or,

$$\boxed{\frac{d}{dt}\left(\frac{\zeta+f}{dp}\right) = 0}$$

The absolute vorticity divided by the pressure thickness of a layer is called the **potential vorticity**. Thus, potential vorticity is conserved.



The most common application of potential vorticity is an eastward moving air mass, in the mid-latitudes, that encounters a north-south mountain barrier. As the air mass approaches the barrier, its depth must decrease because the stable stratosphere resists vertical movement. According to conservation of potential vorticity, the absolute vorticity will also decrease. Since the air mass is initially moving east, the Earth's vorticity is not changing. Thus, the relative vorticity must decrease, which forms an upstream ridge and gradually turns the air mass towards the south.

After crossing the barrier crest, the air mass has a component towards the south, which leads to a decrease in the Earth's vorticity. The depth of the air mass is now increasing, as should the absolute vorticity according to conservation of potential vorticity. Because the Earth's vorticity is decreasing, the relative vorticity must increase very strongly. Thus, a downstream trough is formed.

As relative vorticity increases downstream of the barrier, the air mass is gradually turned towards the north. Since the depth of the air mass is no longer changing, further movement should conserve absolute vorticity, with alternating ridges and troughs until friction erodes the flow.