Vector Derivation of Large-scale Vorticity Equation

Start with the large-scale equation of motion using pressure as the vertical co-ordinate,

$$\frac{d\vec{V}}{dt} = -f\vec{k}\times\vec{V} - \nabla_p\phi.$$

Expand the total derivative,

$$\frac{\partial\vec{V}}{\partial t} + \vec{V}\cdot\nabla_p\vec{V} + \omega\frac{\partial\vec{V}}{\partial p} = -f\vec{k}\times\vec{V} - \nabla_p\phi.$$

Label each term: A B C D E

Next, operate on each term (A through E) with $\vec{k}\cdot\nabla_p\times$

Term A:

$$\vec{k}\cdot\nabla_p\times\left(\frac{\partial\vec{V}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\vec{k}\cdot\nabla_p\times\vec{V}\right) = \frac{\partial\zeta}{\partial t}.$$

Term B:

Use the identity,

$$\left(\vec{V}\cdot\nabla_p\right)\vec{V} = \frac{1}{2}\nabla_p\left(\vec{V}\cdot\vec{V}\right) + \zeta\vec{k}\times\vec{V}.$$

So,

$$\vec{k}\cdot\nabla_p\times\left(term\ B\right) = \frac{1}{2}\vec{k}\cdot\nabla_p\times\left[\nabla_p\left(\vec{V}\cdot\vec{V}\right)\right] + \vec{k}\cdot\nabla_p\times\left(\zeta\vec{k}\times\vec{V}\right).$$

Use the identity,

$$\nabla\times\nabla A = 0,$$

and,

$$\vec{k}\cdot\nabla_p\times\left(term\ B\right) = \vec{k}\cdot\nabla_p\times\left(\zeta\vec{k}\times\vec{V}\right).$$

Use another identity,

$$\nabla\times\left(\vec{A}\times\vec{B}\right) = \vec{A}\left(\nabla\cdot\vec{B}\right) - \vec{B}\left(\nabla\cdot\vec{A}\right) - \left(\vec{A}\cdot\nabla\right)\vec{B} + \left(\vec{B}\cdot\nabla\right)\vec{A},$$

and,

$$\nabla_p\times\left(\zeta\vec{k}\times\vec{V}\right) = \zeta\vec{k}\left(\nabla_p\cdot\vec{V}\right) - \vec{V}\left(\nabla_p\cdot\zeta\vec{k}\right) - \left(\zeta\vec{k}\cdot\nabla_p\right)\vec{V} + \left(\vec{V}\cdot\nabla_p\right)\zeta\vec{k}.$$

Or,

$$\nabla_p\times\left(\zeta\vec{k}\times\vec{V}\right) = \vec{k}\zeta\left(\nabla_p\cdot\vec{V}\right) - \vec{V}\left(\nabla_p\cdot\zeta\vec{k}\right) - \left(\zeta\vec{k}\cdot\nabla_p\right)\vec{V} + \left(\vec{V}\cdot\nabla_p\zeta\right)\vec{k}.$$

Do a $\vec{k}$ dot both sides of this equation,

$$\vec{k}\cdot\nabla_p\times\left(\zeta\,\vec{k}\times\vec{V}\right) = \vec{k}\cdot\vec{k}\zeta\left(\nabla_p\cdot\vec{V}\right) - \vec{k}\cdot\vec{V}\left(\nabla_p\cdot\zeta\,\vec{k}\right) -$$
$$\left(\zeta\,\vec{k}\cdot\nabla_p\right)\vec{k}\cdot\vec{V} + \left(\vec{V}\cdot\nabla_p\zeta\right)\vec{k}\cdot\vec{k} \quad .$$

Since

$$\vec{k}\cdot\vec{V} = 0,$$

$$\vec{k}\cdot\nabla_p\times\left(\zeta\,\vec{k}\times\vec{V}\right) = \zeta\left(\nabla_p\cdot\vec{V}\right) + \left(\vec{V}\cdot\nabla_p\zeta\right).$$

Term C:
Use the distributive law,

$$\vec{k}\cdot\nabla_p\times\left(\omega\frac{\partial\vec{V}}{\partial p}\right) = \vec{k}\cdot\left[\nabla_p\omega\times\frac{\partial\vec{V}}{\partial p} + \omega\nabla_p\times\frac{\partial\vec{V}}{\partial p}\right].$$

Or,

$$\vec{k}\cdot\nabla_p\times\left(\omega\frac{\partial\vec{V}}{\partial p}\right) = \vec{k}\cdot\left[\nabla_p\omega\times\frac{\partial\vec{V}}{\partial p}\right] + \vec{k}\cdot\left[\omega\nabla_p\times\frac{\partial\vec{V}}{\partial p}\right].$$

Changing the order of the partial differentiation in the last term,

$$\vec{k}\cdot\nabla_p\times\left(\omega\frac{\partial\vec{V}}{\partial p}\right) = \vec{k}\cdot\left[\nabla_p\omega\times\frac{\partial\vec{V}}{\partial p}\right] + \omega\frac{\partial}{\partial p}\left(\vec{k}\cdot\nabla_p\times\vec{V}\right).$$

Or,

$$\vec{k}\cdot\nabla_p\times\left(\omega\frac{\partial\vec{V}}{\partial p}\right) = \vec{k}\cdot\left[\nabla_p\omega\times\frac{\partial\vec{V}}{\partial p}\right] + \omega\frac{\partial\zeta}{\partial p}.$$

Term D:
Use an identity,

$$\vec{k}\cdot\nabla_p\times\left(-f\,\vec{k}\times\vec{V}\right) =$$
$$-\vec{k}\cdot\left[f\,\vec{k}\left(\nabla_p\cdot\vec{V}\right) - \vec{V}\left(\nabla_p\cdot f\,\vec{k}\right) - \left(f\,\vec{k}\cdot\nabla_p\right)\vec{V} + \left(\vec{V}\cdot\nabla_p\right)f\,\vec{k}\right].$$

Or,

$$\vec{k}\cdot\nabla_p\times\left(-f\,\vec{k}\times\vec{V}\right) =$$
$$-\left[\vec{k}\cdot f\,\vec{k}\left(\nabla_p\cdot\vec{V}\right) - \vec{k}\cdot\vec{V}\left(\nabla_p\cdot f\,\vec{k}\right) - \left(f\,\vec{k}\cdot\nabla_p\right)\vec{k}\cdot\vec{V} + \left(\vec{V}\cdot\nabla_p\right)\vec{k}\cdot f\,\vec{k}\right].$$

Or,

$$\vec{k}\cdot\nabla_p\times\left(-f\,\vec{k}\times\vec{V}\right) = -f\left(\nabla_p\cdot\vec{V}\right) - \left(\vec{V}\cdot\nabla_p\right)f\,.$$

Term E:

$$\vec{k} \cdot \nabla_p \times \left(-\nabla_p \phi\right) = 0,$$

using an identity.

Putting all the terms together,

$$\frac{\partial \zeta}{\partial t} + \zeta\left(\nabla_p \cdot \vec{V}\right) + \left(\vec{V} \cdot \nabla_p \zeta\right) + \vec{k} \cdot \left[\nabla_p \omega \times \frac{\partial \vec{V}}{\partial p}\right] + \omega \frac{\partial \zeta}{\partial p} =$$

$$- f\left(\nabla_p \cdot \vec{V}\right) - \left(\vec{V} \cdot \nabla_p\right) f,$$

or,

$$\frac{\partial \zeta}{\partial t} + \left(\vec{V} \cdot \nabla_p \zeta\right) + \omega \frac{\partial \zeta}{\partial p} + \left(\vec{V} \cdot \nabla_p\right) f =$$

$$- \zeta\left(\nabla_p \cdot \vec{V}\right) - f\left(\nabla_p \cdot \vec{V}\right) - \vec{k} \cdot \left[\nabla_p \omega \times \frac{\partial \vec{V}}{\partial p}\right],$$

or,

$$\frac{d\zeta}{dt} + \left(\vec{V} \cdot \nabla_p\right) f = - \zeta\left(\nabla_p \cdot \vec{V}\right) - f\left(\nabla_p \cdot \vec{V}\right) - \vec{k} \cdot \left[\nabla_p \omega \times \frac{\partial \vec{V}}{\partial p}\right].$$

Since f does not change with time or p,

$$\frac{d\left(\zeta + f\right)}{dt} = -\left(\zeta + f\right)\left(\nabla_p \cdot \vec{V}\right) - \vec{k} \cdot \left[\nabla_p \omega \times \frac{\partial \vec{V}}{\partial p}\right].$$

Finally, the vorticity equation is scaled for synoptic-scale circulations. The twisting term, the product of relative vorticity and horizontal divergence, and the vertical advection of absolute vorticity are neglected. Thus, the **scaled vorticity equation** is

$$\boxed{\frac{d_p\left(\zeta + f\right)}{dt} = - f\left(\nabla_p \cdot \vec{V}\right)} \; .$$