Vector Derivation of the Thermal Wind

Start with the equation for the geostrophic wind using pressure as the vertical co-ordinate,

$$f_o \vec{V}_g = \vec{k} \times \nabla_p \phi \,.$$

Take the partial derivative with respect to pressure,

$$f_o \frac{\partial \vec{V}_g}{\partial p} = \vec{k} \times \nabla_p \frac{\partial \phi}{\partial p} . \qquad (1)$$

The partial derivative of the geopotential with respect to pressure can be replaced from the hydrostatic equation,

$$\frac{\partial \phi}{\partial p} = -\alpha .$$

Substitute for the specific volume from the equation of state

$$\alpha = \frac{RT}{p} ,$$

$$\frac{\partial \phi}{\partial p} = -\frac{RT}{p} .$$

Substitute into equation (1),

$$f_o \frac{\partial \vec{V}_g}{\partial p} = -\vec{k} \times \nabla_p \frac{RT}{p} .$$

On an isobaric (constant pressure) surface,

$$f_o \frac{\partial \vec{V}_g}{\partial p} = -\frac{R}{p} \vec{k} \times \nabla_p \overline{T} .$$

Next, multiply both sides by dp and assume that the geostrophic wind changes more rapidly with pressure than any other independent variable,

$$f_o \frac{\partial \vec{V}_g}{\partial p} dp = -\frac{R}{p} dp \, \vec{k} \times \nabla_p \overline{T} ,$$

and,

$$f_o \, d\vec{V}_g = -R \, d \ln p \, \vec{k} \times \nabla_p \overline{T} ,$$

or,

$$-d\vec{V}_g = \frac{R}{f_o} \, d\ln p \; \vec{k} \times \nabla_p \overline{T} \; .$$

Finally, the thermal wind is defined as the change in wind, upper minus lower,

$$\boxed{\vec{V}_T = \frac{R}{f_o} \, d\ln p \; \vec{k} \times \nabla_p \overline{T}} \; .$$

Thus, the thermal wind is 90 degrees to the left of the temperature gradient. If one stands with the thermal wind at their back, cold air is on the left.