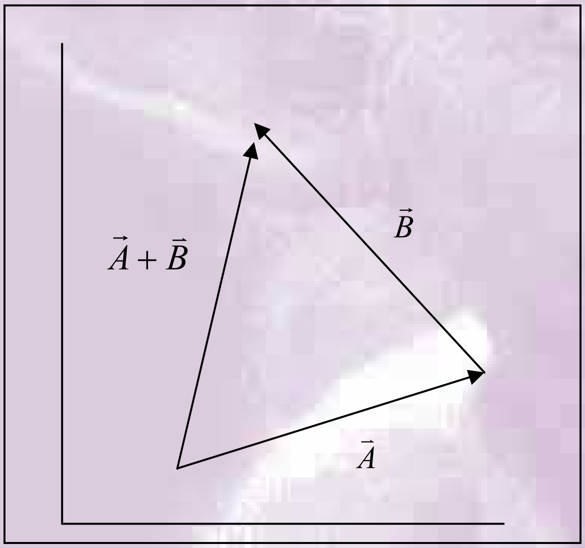
Vector Basics

A quantity that has only magnitude is called a **scalar**. Examples are temperature and pressure. A quantity that has both magnitude and direction is called a **vector**. A few examples are velocity, acceleration, and force.

In books and papers, many different methods are used to denote quantities as vectors according to the author's preference. Common choices include bold type, as well as arrows and carrots above the symbol. I prefer an arrow over a letter, for example, $\vec{A}$.

Since a vector has two parts, magnitude and direction, it is often expressed as the product of a scalar magnitude and a **unit vector**, which has a magnitude of one, for example $A\vec{i}$. $\frac{\vec{A}}{|A|}$ is also a unit vector in the direction of the vector $\vec{A}$ because its magnitude is one. However, the most commonly used unit vectors are orientated along each coordinate axis, and away from the origin. In an x, y, z co-ordinate system the usual unit vectors are $\vec{i}\ \vec{j}\ and\ \vec{k}$, respectively. These unit vectors point in the direction in which position (x, y, or z) increases. This also defines a **positive direction**. To indicate a **negative direction**, a minus sign is placed in front of the magnitude. Remember that magnitude is always positive, and if a minus sign is present it indicates direction.

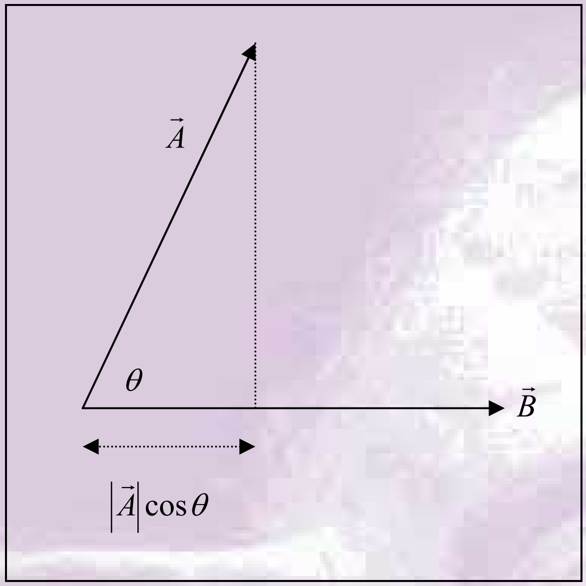


Multiplication of a scalar and a vector follows the basic rules of algebra. When performing arithmetic with two vectors, special vector rules must be used.

The physical meaning of **adding two vectors** is shown above. By placing the tail of the second vector at the head of the first (preserving direction and magnitude), the sum is clear. Mathematically, a vector sum is usually computed using components. If

$\vec{A}=a_x\vec{i}+a_y\vec{j}$ and $\vec{B}=b_x\vec{i}+b_y\vec{j}$, then $\vec{A}+\vec{B}=(a_x+b_x)\ \vec{i}+(a_y+b_y)\ \vec{j}$.

Multiplying two vectors is more involved because there are several different types of vector multiplication. Each has a different physical meaning. A multiplication that one might expect to be the most straight forward, $\vec{A}\vec{B}$, is actually the most complex. This type results in a **tensor**, which is a vector whose components are vectors. Tensors occur in meteorology when studying friction, but are usually not used in undergraduate study. Two types of vector multiplication which are widely used in undergraduate meteorology are the dot and cross products.



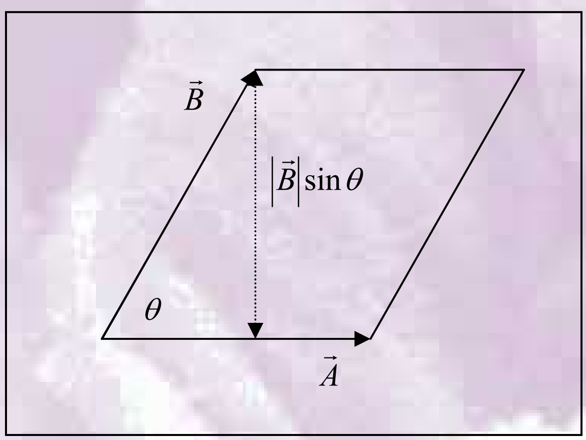
The **dot product** of two vectors is the product of the two magnitudes and the cosine of the angle between the vectors, $\vec{A}\cdot\vec{B} = |\vec{A}|\ |\vec{B}|\ \cos\theta$, which is a scalar.

Thus, the dot product is large and positive when the two vectors are in the same direction, zero when they are perpendicular, and large but negative when they are in opposite directions.

Common physical interpretations of the dot product are: it measures the degree to which one vector is in the same direction as another vector; and it is the projection of one vector onto another.

To find the component of a vector along one of the coordinate axes, dot the vector with the appropriate unit vector. For example, if $\vec{A}=a_x\vec{i}+a_y\vec{j}$ and one wants the component along the x axis, find $\vec{i}\cdot\vec{A}=a_x\vec{i}\cdot\vec{i}+a_y\vec{i}\cdot\vec{j}=a_x(1)+a_y(0)=a_x$.

The **cross product** of two vectors is a vector that is perpendicular to both vectors. In the diagram below, $\vec{A}$ crossed into $\vec{B}$ would be out of the page. $\vec{B}$ crossed into $\vec{A}$ would be into the page. The magnitude of the cross product is the product of the two magnitudes and the sine of the angle between the vectors, $|\vec{A}\times\vec{B}| = |\vec{A}|\ |\vec{B}|\ \sin\theta$, where the positive direction for θ is counter-clockwise from the first vector towards the second.



A common physical interpretation of the cross product is that it represents the area of a parallelogram formed by the two vectors. If the two vectors are parallel, then the cross product is zero.

When the unit vector $\vec{k}$ is crossed with a horizontal vector, the result is a vector with the same magnitude that has been rotated 90 degrees counter-clockwise. If this is done a second time, the result is a vector pointing in the opposite direction of the initial vector.

$$\vec{k} \times \left(\vec{k} \times \vec{A}\right) = -\vec{A}$$

Generally, **vector differentiation** is similar to ordinary differentiation. Think of a vector as a product of a scalar magnitude and a unit vector. Then use the distributive law as appropriate. Also keep in mind that the derivative of some unit vectors is zero, since the derivative of a constant vector is zero. However, for a vector to be constant, both the magnitude and direction must be constant.

For example, in Cartesian coordinates,

$$\text{if } \vec{A} = a_x \vec{i}\text{, then } \frac{d\vec{A}}{dt} = \frac{d}{dt}\left(a_x \vec{i}\right) = a_x \frac{d\vec{i}}{dt} + \frac{da_x}{dt}\vec{i} = \frac{da_x}{dt}\vec{i}\ .$$

However, in polar co-ordinates,

$$\text{if } \vec{A} = a_r \vec{r}\text{, then } \frac{d\vec{A}}{dt} = \frac{d}{dt}\left(a_r \vec{r}\right) = a_r \frac{d\vec{r}}{dt} + \frac{da_r}{dt}\vec{r}\ .$$

Partial derivatives with respect to the spatial co-ordinates are sometimes called **directional derivatives**, because they are changes of a quantity in a particular direction. Again, use the distributive law as appropriate, and keep in mind that the derivative of some unit vectors is zero.

A special vector differential operator, **del**, defined as $\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$,

occurs quite often in scientific applications. Del of a scalar, ϕ, is called the **gradient** of ϕ. The gradient of a scalar is a vector that generally has a component along each co-ordinate axis. The direction of the gradient is always from low to high values and perpendicular to isopleths of the scalar. The three dimensional gradient is sometimes expressed as the sum of a horizontal and a vertical component. The gradient of scalars, such as pressure and temperature, are quite important in meteorology.

For example, if at some time and altitude, $T(x, y) = ay + b\sin kx$, where a, b, and k are constants, then the temperature gradient is

$$\nabla T = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}\right)\left(ay + b\sin kx\right) = bk\cos kx\ \vec{i} + a\ \vec{j}\ ,$$

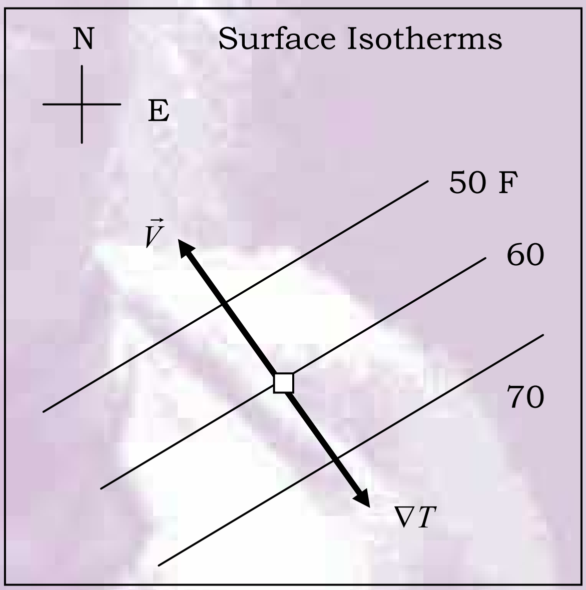
a non-constant vector.

Del can not be dotted with a scalar, since dot multiplication is an operation between two vectors. Del dot a vector, $\vec{A}$, is called the **divergence** of $\vec{A}$. Divergence is a scalar that can also be expressed as the sum of a horizontal and a vertical part.

$$\nabla \cdot \vec{A} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \right) \cdot \vec{A} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} \right) \cdot \vec{A} + \left(\vec{k}\frac{\partial}{\partial z} \right) \cdot \vec{A} = \nabla_H \cdot \vec{A}_H + \frac{\partial a_z}{\partial z}.$$

The divergence of the wind and of fluxes, such as heat and moisture, are very important in meteorology.

Del cross a vector, $\vec{A}$, is called the **curl** of $\vec{A}$. Curl is a vector, and is also very important in meteorology. The curl of the wind vector, which meteorologists call **vorticity,** occurs quite often. In the first semester of dynamic meteorology, vorticity was defined as $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, which is actually just the vertical component of the vorticity vector, $\zeta = \vec{k} \cdot \left(\nabla \times \vec{U} \right)$. Note that, $\vec{U}$ is the three dimensional wind, while $\vec{V}$ is just the horizontal wind.



Another very important quantity in meteorology is **advection**, which occurs when the wind crosses isopleths of some scalar, such as temperature. The definition of horizontal temperature advection is $-\vec{V} \cdot \nabla T$. Thus, the maximum positive horizontal temperature advection occurs when the horizontal wind and horizontal temperature gradient are in opposite directions, as shown in the figure at the left. With this arrangement wind is blowing from warm towards cold. Also note that the formula for advection does not require movement of the isotherms, just that the moving air parcels (wind) cross the isotherms.