Vector Governing Equations

Equation of State

$$\boxed{\alpha \; p = R_d T} \quad .$$

Vector Derivation of the First Law of Thermodynamics

Divide the First Law of Thermodynamics by dt,

$$\frac{dQ}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} .$$

Since,

$$\frac{dp}{dt} \equiv \omega ,$$

$$\frac{dQ}{dt} = c_p \frac{dT}{dt} - \alpha\, \omega .$$

Expand the total derivative of T,

$$\frac{dQ}{dt} = c_p \left(\frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T + \omega \frac{\partial T}{\partial p} \right) - \alpha\, \omega .$$

Rearranging,

$$\frac{1}{c_p} \frac{d\,Q}{d\,t} = \left(\frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T \right) + \omega \frac{\partial T}{\partial p} - \frac{\alpha\, \omega}{c_p} =$$

$$\left(\frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T \right) + \omega \left(\frac{\partial T}{\partial p} - \frac{\alpha}{c_p} \right) \qquad (1)$$

Recall the definition of potential temperature,

$$\theta \equiv T \left(\frac{1000}{p} \right)^{\frac{R_d}{c_p}} .$$

Take the natural log,

$$\ln\theta = \ln T + \left(\frac{R_d}{c_p}\right)\left(\ln 1000 - \ln p\right).$$

Take the partial derivative with respect to p,

$$\frac{\partial \ln\theta}{\partial p} = \frac{\partial \ln T}{\partial p} - \left(\frac{R_d}{c_p}\right)\frac{\partial \ln p}{\partial p}.$$

Or,

$$\frac{\partial \ln\theta}{\partial p} = \frac{1}{T}\frac{\partial T}{\partial p} - \left(\frac{R_d}{c_p}\right)\frac{1}{p}\frac{\partial p}{\partial p}.$$

Multiply both sides by T,

$$T\frac{\partial \ln\theta}{\partial p} = \frac{\partial T}{\partial p} - \left(\frac{R_d}{c_p}\right)\frac{T}{p}.$$

Using the equation of state in the last term on the RHS,

$$T\frac{\partial \ln\theta}{\partial p} = \frac{\partial T}{\partial p} - \left(\frac{\alpha}{c_p}\right).$$

Substituting into equation (1),

$$\boxed{\frac{\partial T}{\partial t} + \vec{V}\cdot\nabla T + \omega T\frac{\partial \ln\theta}{\partial p} = \frac{1}{c_p}\frac{dQ}{dt}}.$$

Vector Derivation of the Continuity Equation

Using pressure as the vertical co-ordinate and components, the Continuity Equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0,$$

$$\boxed{\nabla_p\cdot\vec{V} + \frac{\partial \omega}{\partial p} = 0}$$ using vectors.