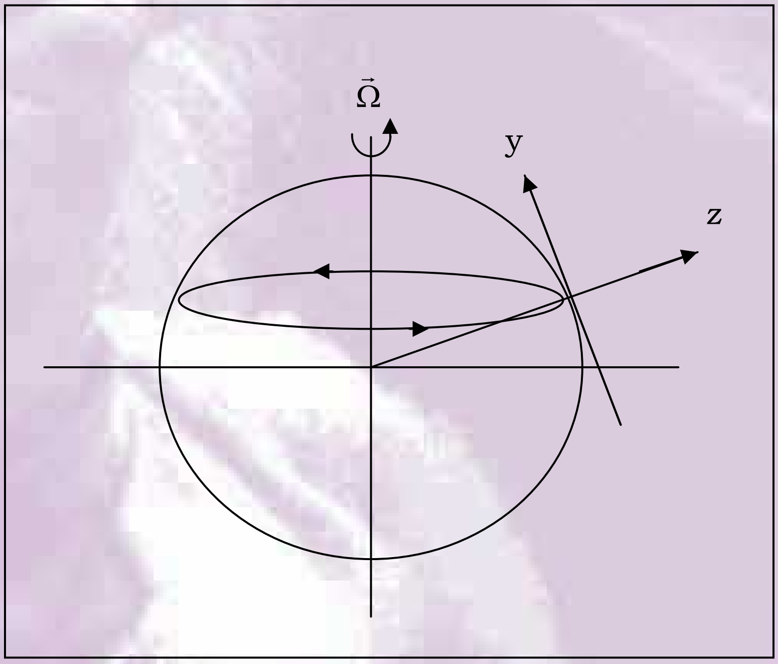
Vector Derivation of the Equation of Motion

The Equation of Motion is the meteorological form of **Newton's Second Law**. Thus, it is based upon a fundamental law of physics, which states that the acceleration (change of velocity with time) of an object in an absolute reference frame is equal to the sum of the forces per unit mass acting on that object. The possible, or **Newtonian, forces** are the pressure gradient force, gravitational force, and frictional force.

$$\frac{d_a\vec{U}_a}{dt} = \sum\frac{\vec{F}}{m} = P\vec{G}F - \vec{G} - \vec{F} \qquad (1)$$

The subscript, a, denotes a quantity measured in an absolute reference frame. And $\vec{U}_a$ is the velocity, measured relative to the origin of an absolute reference frame.



The acceleration is a considerable problem for Meteorologists, who do not work in an absolute reference frame. Instead, they use a Cartesian north-south east-west co-ordinate system that is tangent to the surface of the Earth. As the Earth rotates about its axis with angular velocity, $\vec{\Omega}$, the coordinate origin moves through a circular path, and the coordinate axes rotate about the origin. Thus, the total derivative in an absolute reference frame needs to be related to that in a relative reference frame. The needed relationship for any vector, $\vec{A}$, is

$$\frac{d_a\vec{A}}{dt} = \frac{d\vec{A}}{dt} + \vec{\Omega}\times\vec{A}. \qquad (2)$$

First, apply this formula is to the **position vector**, $\vec{r}$,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

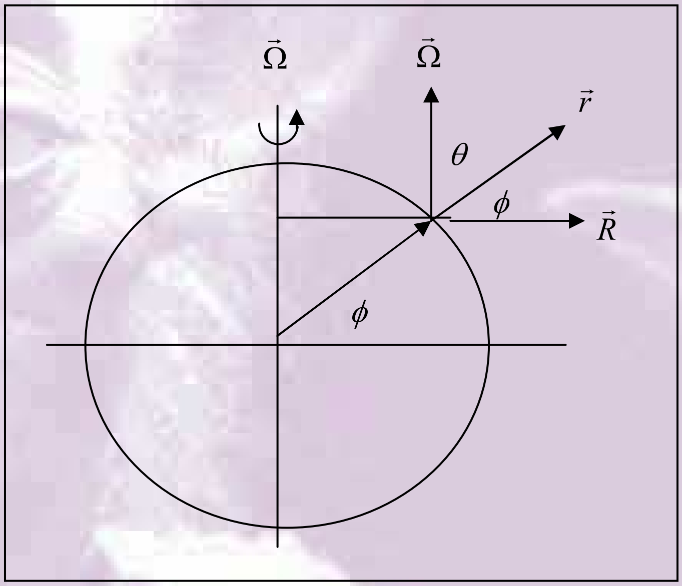
It extends from the Earth's center to any point with co-ordinates (x,y,z).

Substituting $\vec{r}$ for $\vec{A}$ in equation (2),

$$\frac{d_a\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r}\,.$$

Since the change of position with time is velocity,

$$\frac{d_a\vec{r}}{dt} = \vec{U}_a \text{ and } \frac{d\,\vec{r}}{dt} = \vec{U}, \text{ where } \vec{U} = u\vec{i} + v\vec{j} + w\vec{k}\,. \qquad (3)$$



Thus,

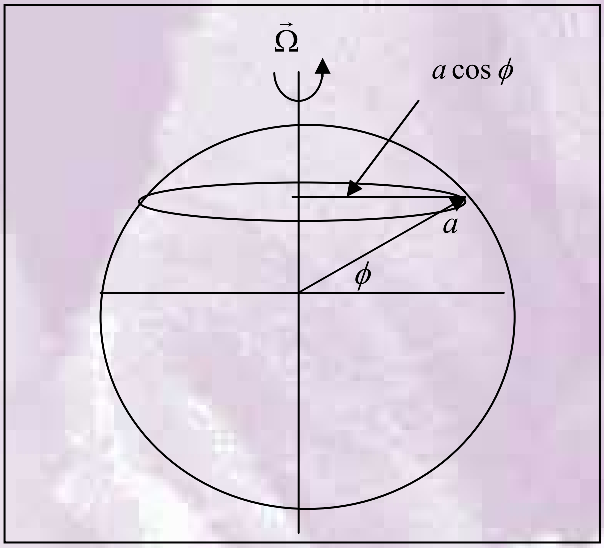
$$\vec{U}_a = \vec{U} + \vec{\Omega}\times\vec{r}\,. \qquad (4)$$

The last term in equation (4) points east according to the right hand rule. The magnitude is given by $\left|\vec{\Omega}\times\vec{r}\right| = \left|\vec{\Omega}\right|\left|\vec{r}\right|\sin\theta = \left|\vec{\Omega}\right|\left|\vec{r}\right|\cos\phi$ where θ is the angle between the two vectors, 90 minus the latitude, ϕ.

The distance from the Earth's center to any point in the atmosphere can be written as $\left|\vec{r}\right| = a + z$ where a is the radius of the Earth, and z is altitude above sea level. Since a is 6371 km and z is ≤ 11 km in the troposphere, $a >> z$, and $\left|\vec{r}\right| \cong a$. Therefore,

$$\left|\vec{\Omega}\times\vec{r}\right| = \left|\vec{\Omega}\right|\left|\vec{r}\right|\cos\phi = \Omega a\cos\phi, \text{ and}$$

$$\vec{\Omega}\times\vec{r} = \Omega a\cos\phi\;\vec{i}\,. \qquad (5)$$



The quantity, $a\cos\phi$, is the distance from the Earth's axis to a point on the Earth's surface at latitude ϕ. As the Earth rotates about its axis, this point travels in a circle of radius, $a\cos\phi$. Since radius times the angular velocity is just linear velocity, the last term in equation (4), $\vec{\Omega}\times\vec{r}$, is just the west to east velocity of the Earth's surface at latitude ϕ. And equation (4) states that the absolute velocity is the sum of the velocity relative to the origin and the velocity of the origin.

Apply equation (2) to the absolute velocity vector, $\vec{U}_a$,

$$\frac{d_a\vec{U}_a}{dt} = \frac{d\vec{U}_a}{dt} + \vec{\Omega}\times\vec{U}_a .$$

Substituting for $\vec{U}_a$ from equation (4) on just the RHS,

$$\frac{d_a\vec{U}_a}{dt} = \frac{d}{dt}\left(\vec{U}+\vec{\Omega}\times\vec{r}\right) + \vec{\Omega}\times\left(\vec{U}+\vec{\Omega}\times\vec{r}\right).$$

Using the distributive law, and noting that $\vec{\Omega}$ is a constant vector,

$$\frac{d_a\vec{U}_a}{dt} = \frac{d\vec{U}}{dt} + \vec{\Omega}\times\frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{U} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right).$$

From equations (3) and (5),

$$\frac{d_a\vec{U}_a}{dt} = \frac{d\vec{U}}{dt} + 2\vec{\Omega}\times\vec{U} + \vec{\Omega}\times\left(\Omega a\cos\phi\ \ \vec{i}\right).$$

Since $\vec{\Omega}$ is perpendicular to $\vec{i}$,

$$\frac{d_a\vec{U}_a}{dt} = \frac{d\vec{U}}{dt} + 2\vec{\Omega}\times\vec{U} - \Omega^2\vec{R}. \qquad (6)$$

The direction of $\vec{R}$ is outward and perpendicular to the Earth's axis, and $\left|\vec{R}\right| = a\cos\phi$.

Equation (6) relates acceleration in an absolute reference frame to that in a relative reference frame. These two accelerations differ by two terms, that are often called **apparent forces**; they arise because of the choice of coordinate systems, and are not Newtonian forces.

Combining equations (1) and (6),

$$\frac{d\vec{U}}{dt} = P\vec{G}F - \vec{G} - \vec{F} - 2\vec{\Omega}\times\vec{U} + \Omega^2\vec{R}.$$

The last two terms on the RHS, the apparent forces, are the **Coriolis** and **Centrifugal** forces. Newtonian gravity is always added to the Centrifugal force to form **apparent gravity**, the gravitational force that appears to be acting on our bodies,

$$-\vec{g} = -\vec{G} + \Omega^2\vec{R}.$$

When using pressure as the vertical coordinate, the **pressure gradient force** is minus the gradient of **geopotential**, ϕ,

$$P\vec{G}F = -g\nabla z = -\nabla\phi .$$

Thus,

$$\frac{d\vec{U}}{dt} = -\nabla\phi - \vec{g} - 2\vec{\Omega}\times\vec{U} - \vec{F} . \tag{7}$$

Equation (7) is the equation of motion, but we are not done. When studying synoptic-scale circulations in dynamic meteorology, the **scaled** version of this equation is normally used. The basic idea is to simplify an equation, so it is easier to use, by ignoring relatively small terms. This is done by estimating the order of magnitude of each term for a particular size circulation, and neglecting the terms that are at least one order of magnitude smaller than the other terms.

When scaling the equation of motion for synoptic-scale circulations, the omitted terms include friction, the vertical component of acceleration, and some of the Coriolis terms. The resulting equation is often written as two equations, one for the vertical component, and one for the horizontal component. The vertical component of the equation of motion is just the **hydrostatic equation**,

$$\boxed{0 = -\frac{\partial\phi}{\partial p} - \alpha} . \tag{8}$$

The horizontal part of the equation of motion is

$$\boxed{\frac{d\vec{V}}{dt} = -\nabla_p\phi - f\,\vec{k}\times\vec{V}} \tag{9}$$

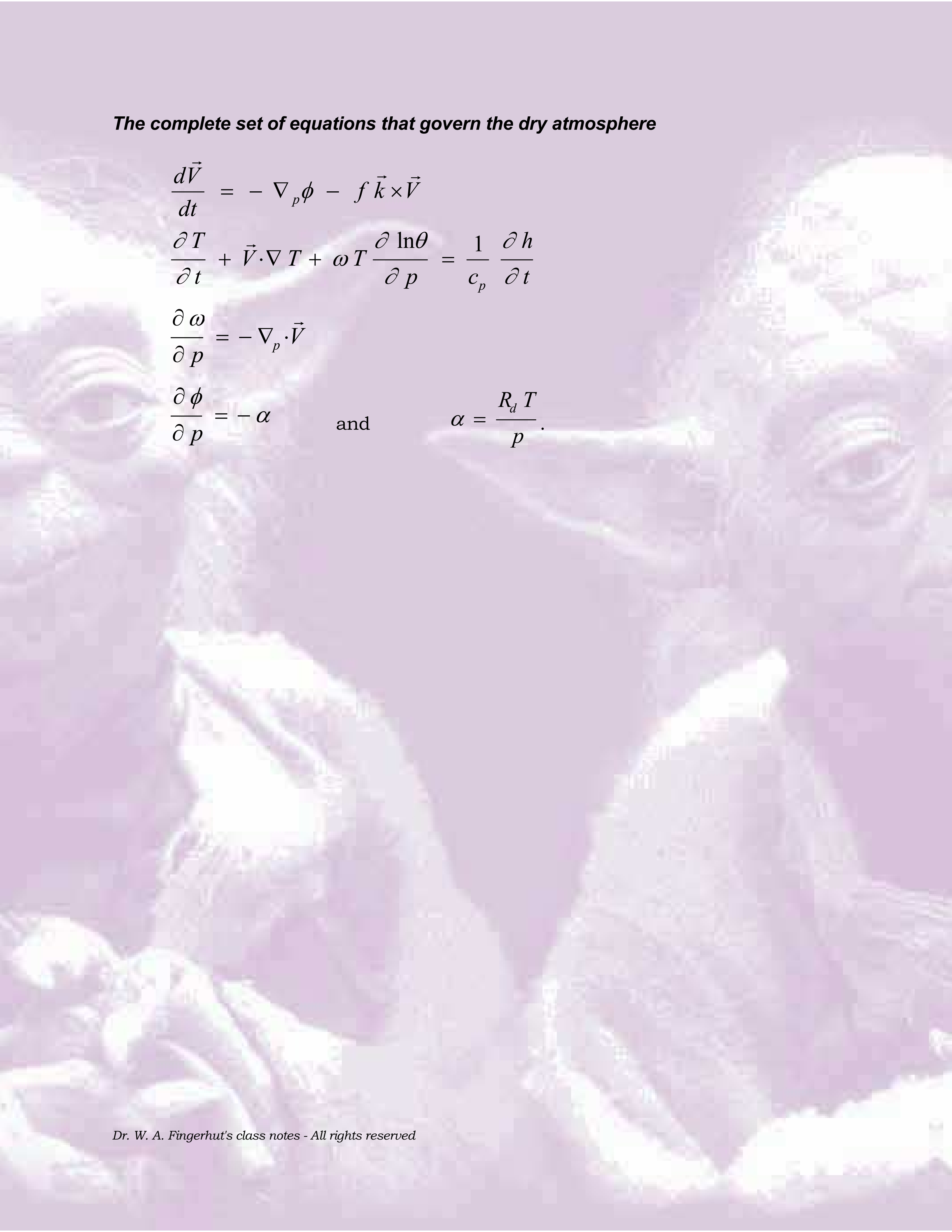
where the differential operator ∇_p does not have a vertical component because it is evaluated along a constant pressure surface. The horizontal velocity is given by

$$\vec{V} = u\vec{i} + v\vec{j} ,$$

and the Coriolis parameter is given by

$$f = 2\Omega \sin\phi .$$

Equations (8) and (9) are the **equation of motion scaled for synoptic-scale circulations**.



The complete set of equations that govern the dry atmosphere

$$\frac{d\vec{V}}{dt} = -\nabla_p \phi - f\,\vec{k}\times\vec{V}$$

$$\frac{\partial T}{\partial t} + \vec{V}\cdot\nabla T + \omega T\,\frac{\partial \ln\theta}{\partial p} = \frac{1}{c_p}\,\frac{\partial h}{\partial t}$$

$$\frac{\partial \omega}{\partial p} = -\nabla_p\cdot\vec{V}$$

$$\frac{\partial \phi}{\partial p} = -\alpha \qquad \text{and} \qquad \alpha = \frac{R_d\,T}{p}.$$