Geostrophic and Ageostrophic Winds

The geostrophic wind is a hypothetical wind that results in a balance between the Coriolis and horizontal pressure gradient forces. To find the geostrophic wind, $\vec{V}_g$, we start with the equation of motion scaled for synoptic-scale circulations,

$$\frac{d\vec{V}}{dt} = -\nabla_p\phi - f\vec{k}\times\vec{V}, \qquad (1)$$

and assume that the acceleration is zero,

$$0 = -\nabla_p\phi - f\vec{k}\times\vec{V}_g. \qquad (2)$$

To isolate the geostrophic wind, perform a $\vec{k}\times$,

$$0 = -\vec{k}\times\nabla_p\phi - \vec{k}\times\left(f\vec{k}\times\vec{V}_g\right).$$

In the last term on the RHS,

$$\vec{V}_g = u_g\vec{i} + v_g\vec{j} \text{ and}$$

$$\vec{k}\times\vec{V}_g = \vec{k}\times\left(u_g\vec{i}+v_g\vec{j}\right) = u_g\vec{j} - v_g\vec{i}.$$

Another $\vec{k}\times$ yields,

$$\vec{k}\times\left(\vec{k}\times\vec{V}_g\right) = \vec{k}\times\left(u_g\vec{j}-v_g\vec{i}\right) = -u_g\vec{i} - v_g\vec{j} = -\vec{V}_g.$$

So,

$$0 = -\vec{k}\times\nabla_p\phi + f\vec{V}_g,$$

or

$$f\vec{V}_g = \vec{k}\times\nabla_p\phi. \qquad (3)$$

The geostrophic wind is especially useful if it has no divergence. Thus, the Coriolis parameter is assumed constant in the definition of the geostrophic wind,

$$\boxed{f_o\vec{V}_g = \vec{k}\times\nabla_p\phi}\ .$$

To find the divergence, a del dot operation is performed on this equation,

$$\nabla_p\cdot f_o\vec{V}_g = \nabla_p\cdot\vec{k}\times\nabla_p\phi.$$

The RHS becomes,

$$\nabla_p \phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} \right) \phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y},$$

and

$$\vec{k} \times \nabla_p \phi = \vec{k} \times \left(\vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} \right) = \left(\vec{j}\frac{\partial \phi}{\partial x} - \vec{i}\frac{\partial \phi}{\partial y} \right),$$

and

$$\nabla \cdot \vec{k} \times \nabla_p \phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} \right) \cdot \left(\vec{j}\frac{\partial \phi}{\partial x} - \vec{i}\frac{\partial \phi}{\partial y} \right) = -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial x \partial y} = 0.$$

Thus,

$$\nabla_p \cdot f_o \vec{V}_g = \nabla_p \cdot \vec{k} \times \nabla_p \phi = 0.$$

When the wind is exactly geostrophic, the Coriolis and pressure gradient forces balance, and there is no acceleration. So, acceleration occurs only when the wind deviates from geostrophic. To better understand this concept, assume the wind is composed of a geostrophic and ageostrophic component,

$$\vec{V} = \vec{V}_g + \vec{V}_a. \qquad (4)$$

Substitute into the RHS of equation (1),

$$\frac{d\vec{V}}{dt} = -\nabla_p \phi - f\vec{k} \times \left(\vec{V}_g + \vec{V}_a \right) = -\nabla_p \phi - f\vec{k} \times \vec{V}_g - f\vec{k} \times \vec{V}_a.$$

According to equation (2), the first two terms on the RHS add up to zero, and

$$\frac{d\vec{V}}{dt} = -f\vec{k} \times \vec{V}_a.$$

Thus, acceleration results from the ageostrophic part of the wind, and is directed 90 degrees to the right of the **ageostrophic wind**.

To solve for the ageostrophic wind, perform a $\vec{k} \times$,

$$\vec{k} \times \frac{d\vec{V}}{dt} = -f\vec{k} \times \left(\vec{k} \times \vec{V}_a \right) = f\vec{V}_a,$$

and

$$\vec{V}_a = f^{-1} \vec{k} \times \frac{d\vec{V}}{dt}.$$

An important cause of ageostrophic wind is height (or pressure) change with time. To find such a relationship, expand the total derivative on the RHS,

$$\vec{V}_a = f^{-1}\vec{k}\times\frac{d\vec{V}}{dt} = f^{-1}\vec{k}\times\left(\frac{\partial \vec{V}}{\partial t} + \vec{U}\cdot\nabla \vec{V}\right).$$

In the local tendency term use equation (4),

$$\vec{V}_a = f^{-1}\vec{k}\times\left(\frac{\partial \vec{V}_g}{\partial t} + \frac{\partial \vec{V}_a}{\partial t} + \vec{U}\cdot\nabla \vec{V}\right).$$

Substitute for the geostrophic wind from equation (3),

$$\vec{V}_a = f^{-1}\vec{k}\times\left(\frac{\partial}{\partial t}\left(f^{-1}\vec{k}\times\nabla_p\phi\right) + \frac{\partial \vec{V}_a}{\partial t} + \vec{U}\cdot\nabla \vec{V}\right).$$

Rearrange,

$$\vec{V}_a = f^{-1}\vec{k}\times\left[\frac{\partial}{\partial t}\left(f^{-1}\vec{k}\times\nabla_p\phi\right)\right] + f^{-1}\vec{k}\times\left(\frac{\partial \vec{V}_a}{\partial t} + \vec{U}\cdot\nabla \vec{V}\right),$$

and

$$\vec{V}_a = f^{-2}\vec{k}\times\left(\vec{k}\times\nabla_p\frac{\partial \phi}{\partial t}\right) + f^{-1}\vec{k}\times\left(\frac{\partial \vec{V}_a}{\partial t} + \vec{U}\cdot\nabla \vec{V}\right),$$

and

$$\vec{V}_a = -f^{-2}\nabla_p\frac{\partial \phi}{\partial t} + f^{-1}\vec{k}\times\left(\frac{\partial \vec{V}_a}{\partial t} + \vec{U}\cdot\nabla \vec{V}\right).$$

The first term on the RHS, which relates height tendency to the ageostrophic wind, is called the **isallobaric wind**. The name comes from a similar expression that contains the pressure tendency when height is used as the vertical co-ordinate.

$$\vec{V}_i = -\frac{1}{f^2}\nabla_p\frac{\partial \phi}{\partial t}$$ or,

$$\vec{V}_i = -\frac{1}{\rho f^2}\nabla_z\frac{\partial p}{\partial t}.$$

Accordingly, a circular region of pressure falls should exhibit an ageostrophic component of the wind which is perpendicular to the isallobars, toward the center of the region.

To derive the divergence of the isallobaric wind, assume that f and ρ are constant and,

$$\nabla \cdot \vec{V}_i = - \frac{1}{\rho_o f_o^2} \nabla \cdot \nabla_z \frac{\partial p}{\partial t} = - \frac{1}{\rho_o f_o^2} \nabla_z^2 \frac{\partial p}{\partial t}.$$

Thus, convergence is expected in a circular region of pressure falls (in addition to geostrophic rotation about low pressure). The converging air must go somewhere, and ascent is the only possibility. If the pressure falls cease, there is no isallobaric wind and associated convergence and ascent.