Trenberth's formulation

The traditional omega equation identifies fundamental processes that determine vertical motion. However, it has the disadvantage of being difficult to evaluate because

$$-\left(\nabla_p^{\,2} + \frac{f_o^2}{\sigma}\,\frac{\partial^2}{\partial p^2}\right)\omega = \left(\sigma^{-1}\nabla_p^{\,2}\right)\left(-\vec{V}_g\cdot\nabla_p\right)\left(-\frac{\partial\phi}{\partial p}\right) +$$

$$\frac{f_o}{\sigma}\left(\frac{\partial}{\partial p}\right)\left[-\vec{V}_g\cdot\nabla_p\left(f_o^{-1}\nabla_p^{\,2}\phi + f\right)\right]$$

the two terms on the RHS tend to partially cancel. (Whenever two large numbers with opposite sign are added the result is small and inaccurate.) To derive another form of the omega equation, expand the two terms on the RHS using the product rule.

First, the temperature advection term, *term B*,

$$\left(\sigma^{-1}\nabla_p^{\,2}\right)\left(-\vec{V}_g\cdot\nabla_p\right)\left(-\frac{\partial\phi}{\partial p}\right) =$$

$$\sigma^{-1}\left[\left(-\vec{V}_g\cdot\nabla_p\right)\left[\nabla_p^{\,2}\left(-\frac{\partial\phi}{\partial p}\right)\right] + \left(-\nabla_p^{\,2}\vec{V}_g\cdot\nabla_p\right)\left(-\frac{\partial\phi}{\partial p}\right)\right].$$

The second term on the RHS can be shown to be,

$$\left(-\nabla_p^{\,2}\vec{V}_g\cdot\nabla_p\right)\left(-\frac{\partial\phi}{\partial p}\right) \cong -f_o\left(\frac{\partial\vec{V}_g}{\partial p}\right)\cdot\nabla_p\left(f_o^{-1}\nabla_p^{\,2}\phi\right).$$

So,

$$\left(\sigma^{-1}\nabla_p^{\,2}\right)\left(-\vec{V}_g\cdot\nabla_p\right)\left(-\frac{\partial\phi}{\partial p}\right) =$$

$$\sigma^{-1}\left(-\vec{V}_g\cdot\nabla_p\right)\left[\nabla_p^2\left(-\frac{\partial\phi}{\partial p}\right)\right] - \frac{f_o}{\sigma}\left(\frac{\partial\vec{V}_g}{\partial p}\right)\cdot\nabla_p\left(f_o^{-1}\nabla_p^{\,2}\phi\right).$$

Next, the differential vorticity advection term, *term C*,

$$\frac{f_o}{\sigma}\left(\frac{\partial}{\partial p}\right)\left[-\vec{V}_g\cdot\nabla_p\left(f_o^{-1}\nabla_p^{\,2}\phi + f\right)\right] =$$

$$\frac{f_o}{\sigma}\left[-\vec{V}_g\cdot\nabla_p\left[f_o^{-1}\nabla_p^{\,2}\left(\frac{\partial\phi}{\partial p}\right)\right] - \left(\frac{\partial\vec{V}_g}{\partial p}\right)\cdot\nabla_p\left(f_o^{-1}\nabla_p^{\,2}\phi + f\right)\right]$$

Or,

$$\frac{f_o}{\sigma}\left(\frac{\partial}{\partial p}\right)\left[-\vec{V}_g\cdot\nabla_p\left(f_o^{-1}\nabla_p^{\,2}\phi + f\right)\right] =$$

$$\sigma^{-1}\left[\vec{V}_g\cdot\nabla_p\left[\nabla_p^{\,2}\left(-\frac{\partial\phi}{\partial p}\right)\right] - f_o\left(\frac{\partial\vec{V}_g}{\partial p}\right)\cdot\nabla_p\left(f_o^{-1}\nabla_p^{\,2}\phi + f\right)\right]$$

Substituting both results into the original omega equation,

$$-\left(\nabla_p^{\,2} + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega =$$

$$\sigma^{-1}\left(-\vec{V}_g\cdot\nabla_p\right)\left[\nabla_p^2\left(-\frac{\partial\phi}{\partial p}\right)\right] - \frac{f_o}{\sigma}\left(\frac{\partial\vec{V}_g}{\partial p}\right)\cdot\nabla_p\left(f_o^{-1}\nabla_p^{\,2}\phi\right) + .$$

$$\sigma^{-1}\left[\vec{V}_g\cdot\nabla_p\left[\nabla_p^{\,2}\left(-\frac{\partial\phi}{\partial p}\right)\right] - f_o\left(\frac{\partial\vec{V}_g}{\partial p}\right)\cdot\nabla_p\left(f_o^{-1}\nabla_p^{\,2}\phi + f\right)\right]$$

The advection of the Laplacian of the temperature terms cancel, and

$$-\left(\nabla_p^{\,2} + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{f_o}{\sigma}\left(\frac{\partial\vec{V}_g}{\partial p}\right)\cdot\nabla_p\left(2f_o^{-1}\nabla_p^{\,2}\phi + f\right).$$

The LHS is proportional to omega. The RHS includes the advection of both geostrophic vorticity and the Earth's vorticity by the thermal wind. Positive vorticity advection corresponds to ascent (negative omega).

The thermal wind is usually from the west, while the gradient of the Earth's vorticity is always north south. As a result, the advection of the Earth's vorticity by the thermal wind is usually small, and sometimes it is neglected.

$$-\left(\nabla_p^{\,2} + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{2f_o}{\sigma}\left(\frac{\partial\vec{V}_g}{\partial p}\right)\cdot\nabla_p\left(f_o^{-1}\nabla_p^{\,2}\phi\right).$$

Substituting from,

$$\zeta_g \;=\; f_o^{-1}\,\nabla_p^2\,\phi$$

and

$$f_o\,\frac{\partial \vec{V}_g}{\partial\, p} = -\,\frac{R}{\overline{p}}\,\vec{k}\times\nabla_p\overline{T}\,,$$

$$-\left(\nabla_p^{\;2} + \frac{f_o^2}{\sigma}\,\frac{\partial^2}{\partial\, p^2}\right)\omega \;=\; -\,\frac{2}{\sigma}\left(-\,\frac{R}{\overline{p}}\;\vec{k}\times\nabla_p\overline{T}\right)\cdot\,\nabla_p\zeta_g\,.$$

Or,

$$\left(\nabla_p^{\;2} + \frac{f_o^2}{\sigma}\,\frac{\partial^2}{\partial\, p^2}\right)\omega \;=\; \frac{2R}{\sigma\,\overline{p}}\left(-\;\vec{k}\times\nabla_p\overline{T}\right)\cdot\,\nabla_p\zeta_g\,.$$

Thus, vertical motion can be inferred from a fairly straight forward equation that involves temperature and geostrophic vorticity at just one level. Usually, temperature increases southward. Thus, $\vec{k}\times\nabla_p\overline{T}$ is positive. Where the geostrophic vorticity decreases towards the east, the result is PVA, the RHS is positive, and omega is negative (ascent).