The Quasi-geostrophic Omega Equation

To form one equation for ω, operate on the energy equation with $f_o^{-1}\,\nabla_p^{\;2}$, operate on the vorticity equation with $\frac{\partial}{\partial\, p}$, and add.

First, the energy equation,

$$\left(f_o^{-1}\,\nabla_p^{\;2}\right)\frac{\partial}{\partial t}\left(-\frac{\partial\,\phi}{\partial\,p}\right) = \left(f_o^{-1}\,\nabla_p^{\;2}\right)\left(-\,\vec{V}_g\cdot\nabla_p\right)\left(-\frac{\partial\,\phi}{\partial\,p}\right) + \left(f_o^{-1}\,\nabla_p^{\;2}\right)\omega\,\sigma.$$

Assuming a constant stability, and changing the order of the partial differentiation in the first term,

$$-\frac{\partial}{\partial\,t}\left(\frac{\partial}{\partial\,p}\right)\left(f_o^{-1}\,\nabla_p^{\;2}\phi\right) = \left(f_o^{-1}\,\nabla_p^{\;2}\right)\left(-\,\vec{V}_g\cdot\nabla_p\right)\left(-\frac{\partial\,\phi}{\partial\,p}\right) + \left(f_o^{-1}\,\sigma\,\nabla_p^{\;2}\right)\omega.$$

Next, the vorticity equation,

$$\left(\frac{\partial}{\partial\,p}\right)\frac{\partial}{\partial\,t}\left(f_o^{-1}\,\nabla_p^{\;2}\phi\right) = \left(\frac{\partial}{\partial\,p}\right)\left[-\,\vec{V}_g\cdot\nabla_p\left(f_o^{-1}\,\nabla_p^{\;2}\phi + f\right)\right] + \left(\frac{\partial}{\partial\,p}\right) f_o\,\frac{\partial\,\omega}{\partial\,p}.$$

Or,

$$\frac{\partial}{\partial\,t}\left(\frac{\partial}{\partial\,p}\right)\left(f_o^{-1}\,\nabla_p^{\;2}\phi\right) = \left(\frac{\partial}{\partial\,p}\right)\left[-\,\vec{V}_g\cdot\nabla_p\left(f_o^{-1}\,\nabla_p^{\;2}\phi + f\right)\right] + f_o\,\frac{\partial^2\omega}{\partial\,p^2}.$$

Adding the two results, the first two terms cancel and

$$0 = \left(f_o^{-1}\,\nabla_p^{\;2}\right)\left(-\,\vec{V}_g\cdot\nabla_p\right)\left(-\frac{\partial\,\phi}{\partial\,p}\right) + \left(f_o^{-1}\,\sigma\,\nabla_p^{\;2}\right)\omega + \left(\frac{\partial}{\partial\,p}\right)\left[-\,\vec{V}_g\cdot\nabla_p\left(f_o^{-1}\,\nabla_p^{\;2}\phi + f\right)\right] + f_o\,\frac{\partial^2\omega}{\partial\,p^2}.$$

Rearrange so that the terms with omega are on the LHS,

$$-\left(f_o^{-1}\,\sigma\,\nabla_p^{\;2}\right)\omega - f_o\,\frac{\partial^2\omega}{\partial\,p^2} = \left(f_o^{-1}\,\nabla_p^{\;2}\right)\left(-\,\vec{V}_g\cdot\nabla_p\right)\left(-\frac{\partial\,\phi}{\partial\,p}\right) + \left(\frac{\partial}{\partial\,p}\right)\left[-\,\vec{V}_g\cdot\nabla_p\left(f_o^{-1}\,\nabla_p^{\;2}\phi + f\right)\right].$$

Multiply by $\frac{f_o}{\sigma}$ and simplify the LHS,

$$-\left(\nabla_p^{\,2} + \frac{f_o^2}{\sigma}\,\frac{\partial^2}{\partial p^2}\right)\omega = \left(\sigma^{-1}\,\nabla_p^{\,2}\right)\left(-\vec{V}_g\cdot\nabla_p\right)\left(-\frac{\partial\,\phi}{\partial\,p}\right) + \frac{f_o}{\sigma}\left(\frac{\partial}{\partial\,p}\right)\left[-\vec{V}_g\cdot\nabla_p\,(f_o^{-1}\,\nabla_p^{\,2}\phi + f)\right].$$

Labeling each term:

A = B + C

Term A is the three dimensional Laplacian of omega. Because of the leading minus sign, *term A* is proportional to plus omega.

Term B is the Laplacian of the horizontal temperature advection. Thus, *term B* is proportional to the negative of the horizontal temperature advection.

Term C is differential vorticity advection.

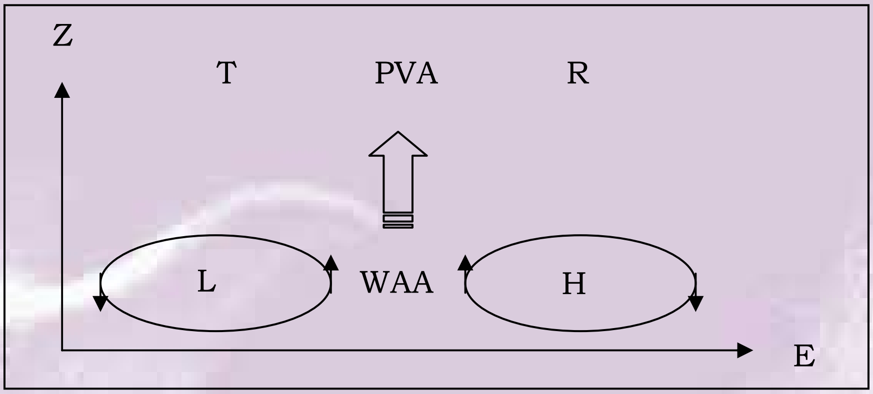
Qualitatively,

$$(omega) \propto -\left(\frac{temperature}{advection}\right) + \left(\frac{vorticity}{advection}\right)_{lower} - \left(\frac{vorticity}{advection}\right)_{upper}.$$

The quasi-geostrophic processes that determine vertical motion are temperature advection and differential vorticity advection. When interpreting the omega equation, remember that omega, the vertical velocity relative to an isobaric surface, is positive during subsidence, and negative during ascent. Therefore, ascent can be caused by warm air advection and positive vorticity advection increasing with height.

In *term C*, the advection of absolute vorticity is small near the ground where the wind is weak and often perpendicular to the vorticity gradient. Since the wind usually increases with height in the troposphere, vorticity advection and its contribution to vertical motion usually increase from the ground to the tropopause.

Temperature advection is usually strongest in the lower troposphere, thus the contribution of temperature advection to vertical motion usually decreases from just above the boundary layer to the tropopause.



In the middle latitudes, where the prevailing winds are from the west, positive vorticity advection and ascent are often found east of a trough axis. (Negative vorticity advection and descent are found west of a trough axis.) The strongest vertical motion occurs when both low level warm air advection and upper level positive vorticity advection are present, which occurs when the trough axis is nearly over the surface low pressure center. The actual location of positive temperature advection varies, as surface pressure centers are found under all parts of upper level waves.