## Quasi-geostrophic Energy Equation

The quasi-geostrophic energy equation can be derived from the First Law of Thermodynamics,

$$\frac{\partial T}{\partial t} + \vec{V}\cdot\nabla_p T + \omega T \frac{\partial \ln\theta}{\partial p} = \frac{1}{c_p}\frac{dQ}{dt}.$$

Replace T from the hydrostatic equation,

$$T = -\frac{p}{R}\frac{\partial \phi}{\partial p},$$

so,

$$\left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla_p + \omega\frac{\partial \ln\theta}{\partial p}\right)\left(-\frac{p}{R}\frac{\partial \phi}{\partial p}\right) = \frac{1}{c_p}\frac{dQ}{dt}.$$

Since p is constant on an isobaric surface,

$$\left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla_p + \omega\frac{\partial \ln\theta}{\partial p}\right)\left(-\frac{\partial \phi}{\partial p}\right) = \frac{R}{p\,c_p}\frac{dQ}{dt}.$$

Introduce the static stability parameter, $\sigma$,

$$\sigma = \frac{\partial \phi}{\partial p}\frac{\partial \ln\theta}{\partial p},$$

$$\left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla_p\right)\left(-\frac{\partial \phi}{\partial p}\right) - \omega\sigma = \frac{R}{p\,c_p}\frac{dQ}{dt}.$$

Assume the flow is adiabatic, and that horizontal advection is done by the geostrophic wind,

$$\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial p}\right) + \left(\vec{V}_g\cdot\nabla_p\right)\left(-\frac{\partial \phi}{\partial p}\right) - \omega\sigma = 0.$$

Or finally,

$$\boxed{\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial p}\right) = -\left(\vec{V}_g\cdot\nabla_p\right)\left(-\frac{\partial \phi}{\partial p}\right) + \omega\sigma}\ .$$

Keeping the hydrostatic equation in mind (the quantity inside the parenthesis is proportional to temperature), the temperature of a layer can be changed by either horizontal temperature advection or vertical motion. Warm air advection and subsidence tend to increase temperature.