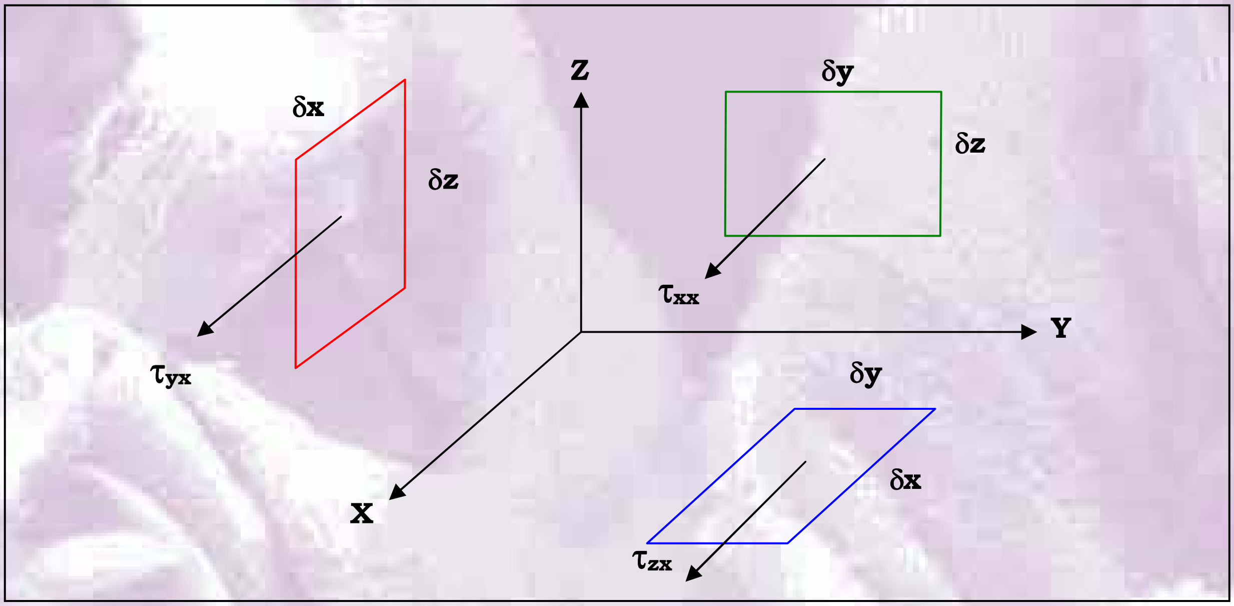
## Molecular or Viscous Friction

Most discussions of friction involve an unfamiliar quantity, the **stress**, $\tau_{ij}$ , which is the force per unit area that keeps a parcel moving with constant velocity. By convention, the subscripts *i* and *j* denote the different components of the stress. The first subscript denotes the co-ordinate axis perpendicular to the area, and the second subscript denotes the component of the force.

If a parcel is considered to be a six sided cube with each side perpendicular to one of the three co-ordinate axes, then there are three distinct areas, each in a plane perpendicular to one of the axes. And since the force is a vector with three components, one along each axis, the stress has nine components, and is a **tensor** (a vector whose components are vectors). For example, three components of the stress would be associated with area in the horizontal plane: the x component of the force yields, $\tau_{zx}$ , the y component of the force yields, $\tau_{zy}$ , and the z component of the force yields, $\tau_{zz}$ .

The frictional force, F, is the net force, associated with the stresses, over all sides of the parcel. The component along the x axis, $F_x$ , is found by considering just the three components of the stress along the x axis, $\tau_{xx}$ , $\tau_{yx}$ , and $\tau_{zx}$ . Each is multiplied by the appropriate area to find the associated force. Then the difference between opposite sides of a parcel is determined for each dimension and summed,

$$F_x = \left[ \frac{\partial \left(\tau_{xx} \ \delta y \ \delta z\right)}{\partial \ x} \delta x + \frac{\partial \left(\tau_{yx} \ \delta x \ \delta z\right)}{\partial \ y} \delta y + \frac{\partial \left(\tau_{zx} \ \delta x \ \delta y\right)}{\partial \ z} \delta z \right].$$



Factor out the volume, $V = \delta x \; \delta y \; \delta z$ ,

$$F_x = V \left( \frac{\partial \; \tau_{xx}}{\partial \; x} + \frac{\partial \; \tau_{yx}}{\partial \; y} + \frac{\partial \; \tau_{zx}}{\partial \; z} \right).$$

Divide by the mass, M, and introduce the density,

$$\frac{F_x}{M} = \frac{1}{\rho} \left( \frac{\partial \; \tau_{xx}}{\partial \; x} + \frac{\partial \; \tau_{yx}}{\partial \; y} + \frac{\partial \; \tau_{zx}}{\partial \; z} \right).$$

Assume that the stress varies more in the vertical than in the horizontal,

$$\frac{F_x}{M} = \frac{1}{\rho} \frac{\partial \; \tau_{zx}}{\partial \; z} . \qquad (1)$$

Equation (1) allows the frictional force per unit mass to be computed. However, the formula contains a quantity, the stress, that is not known or measurable. An approximate formula must be found that involves only known, or mean quantities, which is a procedure known as **parameterization**.

It seems logical to assume that the force per unit area needed to keep a parcel moving with constant velocity (stress) should be proportional to the wind speed and inversely proportional to the distance from the ground. Thus, assume that

$$\tau = \mu \frac{\vec{V}}{z},$$

where $\mu$ is the **dynamic coefficient of viscosity**. If the wind speed is zero at the ground, then

$$\vec{V} = \frac{\partial \; \vec{V}}{\partial \; z} \; z ,$$

and

$$\tau = \mu \frac{\partial \; \vec{V}}{\partial \; z} . \qquad (2)$$

Using equation (2) to parameterize the stress in equation (1),

$$\frac{F_x}{M} = \frac{1}{\rho} \frac{\partial \; \tau_{zx}}{\partial \; z} = \frac{1}{\rho} \frac{\partial}{\partial \; z} \left( \mu \frac{\partial \; u}{\partial \; z} \right) .$$

$$\boxed{\frac{F_x}{M} = \nu \frac{\partial^2 \; u}{\partial \; z^2}} \qquad (3)$$

If $\mu$ is assumed not to change with altitude, and $\nu$ is the **kinematic coefficient of viscosity**.