Mechanical and Total Energy Equations

Begin with the unscaled Equation of motion using altitude as the vertical co-ordinate,

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - 2\,\Omega\, w \cos\phi \qquad (1)$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \qquad (2)$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\,\Omega\, u \cos\phi. \qquad (3)$$

Multiply equation (1) by u, (2) by v, and (3) by w; then add the results together.

$$u\frac{du}{dt} + v\frac{dv}{dt} + w\frac{dw}{dt} = -\frac{1}{\rho}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) - wg -$$

$$2\,\Omega wu \cos\phi + 2\,\Omega wu \cos\phi$$

On the left hand side, LHS, each term can be rewritten using the rule,

$$x\,dx = d\left(\frac{x^2}{2}\right).$$

Thus,

$$\frac{d}{dt}\left(\frac{u^2+v^2+w^2}{2}\right) = -\frac{1}{\rho}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) - wg.$$

Or,

$$\frac{d}{dt}\left(\frac{u^2+v^2+w^2}{2}\right) = -\frac{1}{\rho}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right) - \frac{1}{\rho}\, w\frac{\partial p}{\partial z} - wg.$$

Using the hydrostatic equation,

$$\frac{\partial p}{\partial z} = -g\rho,$$

$$\frac{d}{dt}\left(\frac{u^2+v^2+w^2}{2}\right) = -\frac{1}{\rho}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right) + \frac{1}{\rho}\ wg\rho - wg.$$

Or,

$$\boxed{\frac{d}{dt}\left(\frac{u^2+v^2+w^2}{2}\right) = -\frac{1}{\rho}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right)} \quad . \qquad (5)$$

Equation (5) is the **Mechanical Energy Equation**, which states that parcel kinetic energy is conserved unless parcels move across the isobars. Also, when isobars are crossed from high towards low pressure, parcel kinetic energy increases.

To include thermodynamic effects, recall the First Law of Thermodynamics,

$$dh = c_p\, dT - \alpha\, dp.$$

Substitute from the approximate hydrostatic equation,

$$-\alpha\, dp = g\, dz$$

and divide by dt,

$$\frac{dh}{dt} = c_p \frac{dT}{dt} + g \frac{dz}{dt}.$$

Or,

$$\frac{dh}{dt} = \frac{d}{dt}\left(c_p\, T + g\, z\right) \qquad (6)$$

Thus, parcels conserve their **dry static energy**, $c_p T + g z$, when there is no diabatic heating.

After adding equations (5) and (6),

$$\boxed{\frac{d}{dt}\left(\frac{u^2+v^2+w^2}{2} + g\,z + c_p\,T\right) = -\frac{1}{\rho}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right) + \frac{dh}{dt}} \qquad (7)$$

Equation (7) is the **Total Energy Equation**. Accordingly, an air parcel should conserve the sum of kinetic and dry static energy when there is no flow across the isobars and no diabatic heating.