Ekman Pumping

Ascent will occur at the top of the Ekman layer when the flow across the isobars within the Ekman layer is convergent. Start with the equation of continuity for an incompressible atmosphere,

$$\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0.$$

Neglect the divergence of the u component, and multiply by dz,

$$\frac{\partial w}{\partial z}\, dz = -\,\frac{\partial v}{\partial y}\, dz.$$

Assume that w changes more with z than with other independent variables,

$$dw = -\,\frac{\partial v}{\partial y}\, dz.$$

Integrate over the depth of the Ekman layer,

$$\int_0^{w_E} dw = -\int_0^{z_E} \frac{\partial v}{\partial y}\, dz. \qquad (1)$$

To find the divergence of v, recall the equation for v in the Ekman layer,

$$v = e^{-az}\left[v_{ao}\cos az - u_{ao}\sin az\right].$$

Substitute,

$$u_{ao} = u_o - u_g \quad \text{and} \quad v_{ao} = v_o,$$

$$v = e^{-az}\left[v_o\cos az - (u_o - u_g)\sin az\right].$$

Take the partial derivative with respect to y, and assume that the surface wind does not change in the y direction,

$$\frac{\partial v}{\partial y} = \frac{\partial u_g}{\partial y}\, e^{-az}\sin az.$$

And, since,

$$\varsigma_g = -\,\frac{\partial u_g}{\partial y},$$

$$\frac{\partial v}{\partial y} = -\varsigma_g e^{-az} \sin az . \qquad (2)$$

Substitute equation (2) into (1),

$$\int_0^{w_E} dw = \varsigma_g \int_0^{z_E} e^{-az} \sin az \, dz .$$

Integrate,

$$[w]_0^{w_E} = \varsigma_g \left[-\frac{e^{-az}}{2a} (\sin az + \cos az) \right]_0^{z_E} .$$

Assume the top of the Ekman layer is at π/a, and plug in the limits of integration,

$$w_E = -\frac{\varsigma_g}{2a} \left[e^{-\pi} (\sin\pi + \cos\pi) - e^0 (\sin 0 + \cos 0) \right] .$$

Or,

$$\boxed{w_E = \frac{\varsigma_g}{2a} \left(e^{-\pi} + 1 \right)} . \qquad (3)$$

Thus, ascent should be proportional to positive geostrophic vorticity.

Equation (3) can be used to find the **spin-down time**, the time it takes friction to reduce a circulation by a factor of e^{-1}, τ_e, of a tropospheric circulation. Recall the scaled vorticity equation,

$$\left(\frac{d_p}{dt} \right) (\zeta_g + f) = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) f = f \frac{\partial w}{\partial z} .$$

Assume no change in latitude, and zero vertical velocity at altitude, H, (approximately the tropopause). Since $H >> z_E$,

$$\frac{d_p \zeta_g}{dt} = f \frac{(0 - w_E)}{(H - z_E)} = -f \frac{w_E}{H} .$$

Substitute from equation (3),

$$\frac{d_p \zeta_g}{dt} = -f \frac{\varsigma_g}{2aH} \left(e^{-\pi} + 1 \right) .$$

Or,

$$\frac{d_p \ln\zeta_g}{dt} = -\frac{f}{2aH}\left(e^{-\pi} + 1\right) = -\frac{1}{\tau_e}. \qquad (4)$$

Integrating with respect to time, t,

$$\ln\zeta_g(t) - \ln\zeta_g(0) = -\frac{1}{\tau_e}(t - 0).$$

Or,

$$\zeta_g(t) = \zeta_g(0)\exp\left(-t/\tau_e\right). \qquad (5)$$

Equation (5) shows that friction reduces the vorticity of a circulation exponentially with time. Since $e^{-\pi} << 1$, equation (4) gives,

$$\tau_e = \frac{2aH}{f}.$$

Substitute for a,

$$\tau_e = H\left(\frac{2\rho}{\mu f}\right)^{1/2}.$$

Using a depth of 10 km, density of 1 kg/m^3, f of 10^{-4} s^{-1}, and μ of 2 kg m^{-1} s^{-1}, the spin-down time is about 11 days.