

## Eddy Flux of Zonal Momentum

To see how eddies transport zonal momentum, obtain the flux form of the zonal component of the equation of motion, then apply the perturbation method.

### Flux Form of Zonal Momentum Equation

Many equations in meteorology contain a total derivative of a meteorological variable. When the total derivative is expanded, the advective terms appear, and the equation is said to be in the **advective form**. The advective form of an equation is not the best form to use in some applications because the advective terms are very hard to evaluate accurately. In numerical weather prediction, for example, the resulting errors lead to poor forecasts.

An alternative form of the equations, that has many advantages, is the **flux form**. To derive the flux form of an equation,

1. Expand the total derivative to reveal the advective terms, and if using height as the vertical co-ordinate multiply by the density;
2. Multiply the equation of continuity by the dependent variable in the first equation;
3. Add the two equations resulting from steps one and two.

To derive the flux form of the east-west component of the equation of motion, using height as the vertical co-ordinate, start with the governing equation,

$$\frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv.$$

Expand the total derivative and multiply by the density,

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \rho fv. \quad (1)$$

Multiply the equation of continuity by  $u$ ,

$$u \frac{\partial \rho}{\partial t} + u \frac{\partial \rho u}{\partial x} + u \frac{\partial \rho v}{\partial y} + u \frac{\partial \rho w}{\partial z} = 0. \quad (2)$$

Add equations (1) and (2), using the product rule,

$$\boxed{\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u v}{\partial y} + \frac{\partial \rho u w}{\partial z} = -\frac{\partial p}{\partial x} + f \rho v} \quad (3)$$

Equation (3) governs changes to the zonal momentum by all scales of motion. To explicitly determine the effect of the eddies, use the perturbation method.

## The Perturbation Method

The governing equations apply to the net sum of all scales of motion. After doing a scale analysis, the governing equations are valid for only one scale of motion. We often want to examine how one scale of motion affects another. To accomplish this goal, one can use the perturbation method.

1. Assume that each dependent variable, except the density, can be expressed as the sum of a mean (with an over bar) and an eddy (with a prime) part. For example,

$$u = \bar{u} + u' \quad \text{and} \quad v = \bar{v} + v'.$$

2. Substitute into the governing equation, equation (3),

$$\frac{\partial \rho (\bar{u} + u')}{\partial t} + \frac{\partial \rho (\bar{u} + u') (\bar{u} + u')}{\partial x} + \frac{\partial \rho (\bar{u} + u') (\bar{v} + v')}{\partial y} + \frac{\partial \rho (\bar{u} + u') (\bar{w} + w')}{\partial z} = - \frac{\partial (\bar{p} + p')}{\partial x} + f \rho (\bar{v} + v')$$

Or,

$$\frac{\partial \rho (\bar{u} + u')}{\partial t} + \frac{\partial \rho (\bar{u}^2 + 2\bar{u}u' + u'u')}{\partial x} + \frac{\partial \rho (\bar{u}\bar{v} + \bar{v}u' + \bar{u}v' + u'v')}{\partial y} + \frac{\partial \rho (\bar{u}\bar{w} + u'\bar{w} + \bar{u}w' + u'w')}{\partial z} = - \frac{\partial (\bar{p} + p')}{\partial x} + f \rho (\bar{v} + v')$$

3. Average this equation by operating on each term with the averaging operator, the over bar. This operation is distributive; the average of an average is just the average; and the average of an eddy quantity is zero.

$$\frac{\partial \rho \bar{u}}{\partial t} + \frac{\partial \rho (\bar{u}^2 + \overline{u'u'})}{\partial x} + \frac{\partial \rho (\overline{\bar{u}v'} + \overline{u'v'})}{\partial y} + \frac{\partial \rho (\overline{\bar{u}w'} + \overline{u'w'})}{\partial z} = - \frac{\partial \bar{p}}{\partial x} + f \rho \bar{v}$$

Or,

$$\boxed{\frac{\partial \rho \bar{u}}{\partial t} + \frac{\partial \rho \bar{u}^2}{\partial x} + \frac{\partial \rho \bar{u}\bar{v}}{\partial y} + \frac{\partial \rho \bar{u}\bar{w}}{\partial z} = - \frac{\partial (\bar{p})}{\partial x} + f \rho \bar{v} - \frac{\partial \rho \overline{u'u'}}{\partial x} - \frac{\partial \rho \overline{u'v'}}{\partial y} - \frac{\partial \rho \overline{u'w'}}{\partial z}}$$

The last equation is the same as equation (3) except that the effects of the eddies are now present in the last three terms on the RHS. Each is an eddy flux divergence along one of the 3 co-ordinate axes.