Derivation of the Circulation Theorem

The circulation, C, around a three dimensional volume bounded by a smooth surface, is defined as,

$$C = \oint (udx + vdy + wdz).$$

To explore how the circulation changes with time, take the total derivative with respect to time. When differentiating the RHS, use the product rule,

$$\frac{dC}{dt} = \oint \left(\frac{du}{dt}\, dx + \frac{dv}{dt}\, dy + \frac{dw}{dt}\, dz \right) +$$

$$\oint \left(u \frac{d}{dt}(dx) + v \frac{d}{dt}(dy) + w \frac{d}{dt}(dz) \right) \qquad (1)$$

Parts of terms in the second integral look like,

$$\frac{d}{dt}(dx) \approx \frac{d}{dt}(x_2 - x_1) = \frac{dx_2}{dt} - \frac{dx_1}{dt} = u_2 - u_1 = du.$$

Thus,

$$\oint (udu + vdv + wdw) = .5 \oint (du^2 + dv^2 + dw^2).$$

Since the line integral of an exact differential is zero, the second line integral in equation (1) is zero, and

$$\frac{dC}{dt} = \oint \left(\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz \right).$$

Next, substitute for the accelerations from the unscaled equation of motion. Ignoring friction, and grouping similar terms together,

$$\frac{dC}{dt} = -\oint \frac{1}{\rho} \left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz \right) +$$

$$\oint 2\Omega \left[(v\sin\phi - w\cos\phi)\, dx - u\sin\phi\, dy + u\cos\phi\, dz \right] - \oint g dz$$

If the change of pressure with time is small compared with changes associated with other independent variables, then the first line integral on the RHS becomes

$$-\oint \frac{dp}{\rho}.$$

The second line integral involves just geometry, albeit a bit involved. It can be shown to be

$$- \ 2\Omega \ \frac{dF}{dt},$$

where F is the area projected on the equatorial plane by a volume of atmosphere.

The third line integral is zero, because it is the line integral of an exact differential.

Thus, the Circulation Theorem is

$$\boxed{\frac{dC}{dt} = -\oint \frac{dp}{\rho} - \ 2\Omega \ \frac{dF}{dt}} \ .$$