Derivation of the Thermal Wind Equation

One of the most basic and important relationships in meteorology is between the vertical change of the geostrophic wind and the horizontal temperature gradient. Start with the equations for the geostrophic wind using pressure as the vertical co-ordinate,

$$u_g = -\frac{g}{f_o}\frac{\partial z}{\partial y} \qquad v_g = \frac{g}{f_o}\frac{\partial z}{\partial x}.$$

Take the partial derivative with respect to pressure,

$$\frac{\partial u_g}{\partial p} = -\frac{g}{f_o}\frac{\partial}{\partial p}\frac{\partial z}{\partial y} \qquad \frac{\partial v_g}{\partial p} = \frac{g}{f_o}\frac{\partial}{\partial p}\frac{\partial z}{\partial x}.$$

Reverse the order of the partial differentiation,

$$\frac{\partial u_g}{\partial p} = -\frac{g}{f_o}\frac{\partial}{\partial y}\frac{\partial z}{\partial p} \qquad \frac{\partial v_g}{\partial p} = \frac{g}{f_o}\frac{\partial}{\partial x}\frac{\partial z}{\partial p}.$$

Substitute for the change of height with respect to pressure from the hydrostatic equation, using the over bar to denote mean values in a layer,

$$\frac{\partial p}{\partial z} = -g\bar{\rho} \quad .$$

Using the equation of state, $\rho = \frac{p}{R\,T}$,

$$\frac{\partial p}{\partial z} = -\frac{g\,\bar{p}}{R\bar{T}} \quad \text{or} \quad \frac{\partial z}{\partial p} = -\frac{R\bar{T}}{g\,\bar{p}}.$$

Thus,

$$\frac{\partial u_g}{\partial p} = -\frac{g}{f_o}\frac{\partial}{\partial y}\left(-\frac{R\bar{T}}{g\,\bar{p}}\right) = \frac{R}{f_o}\frac{\partial}{\partial y}\left(\frac{\bar{T}}{\bar{p}}\right)$$

and

$$\frac{\partial v_g}{\partial p} = \frac{g}{f_o}\frac{\partial}{\partial x}\left(-\frac{R\bar{T}}{g\,\bar{p}}\right) = -\frac{R}{f_o}\frac{\partial}{\partial x}\left(\frac{\bar{T}}{\bar{p}}\right).$$

Since we are working on an isobaric (constant pressure) surface,

$$\frac{\partial u_g}{\partial p} = \frac{R}{f_o\,\bar{p}}\frac{\partial \bar{T}}{\partial y} \qquad \frac{\partial v_g}{\partial p} = -\frac{R}{f_o\,\bar{p}}\frac{\partial \bar{T}}{\partial x}.$$

Next, multiply both sides by dp,

$$\frac{\partial u_g}{\partial p}\,dp = \frac{R}{f_o\,\bar{p}}\frac{\partial \bar{T}}{\partial y}\,dp \qquad \frac{\partial v_g}{\partial p}\,dp = -\frac{R}{f_o\,\bar{p}}\frac{\partial \bar{T}}{\partial x}\,dp.$$

Because it is more accurate, use,

$$\frac{dp}{\bar{p}} = d\ln p,$$

$$\frac{\partial u_g}{\partial p}\,dp = \frac{R}{f_o}\frac{\partial \bar{T}}{\partial y}\,d\ln p \qquad \frac{\partial v_g}{\partial p}\,dp = -\frac{R}{f_o}\frac{\partial \bar{T}}{\partial x}\,d\ln p.$$

Quite often it is appropriate to assume that some meteorological quantity, in this case the geostrophic wind, changes more rapidly with pressure than any other independent variable. From the chain rule,

$$du_g = \frac{\partial u_g}{\partial t}\,dt + \frac{\partial u_g}{\partial x}\,dx + \frac{\partial u_g}{\partial y}\,dy + \frac{\partial u_g}{\partial p}\,dp.$$

So, it is assumed that,

$$du_g \cong \frac{\partial u_g}{\partial p}\,dp.$$

Thus,

$$du_g = \frac{R}{f_o}\frac{\partial \bar{T}}{\partial y}\,d\ln p \qquad dv_g = -\frac{R}{f_o}\frac{\partial \bar{T}}{\partial x}\,d\ln p.$$

Finally, the thermal wind, which represents the **vertical wind shear**, is defined as the upper level wind minus the lower level wind, (so the sign changes). Remember

$$u_T = -\frac{R}{f_o}\frac{\partial \bar{T}}{\partial y}\,d\ln p$$

$$v_T = \frac{R}{f_o}\frac{\partial \bar{T}}{\partial x}\,d\ln p$$

that the difference in the log of the pressure is still lower level minus upper level, which gives a positive result. Thus, if the layer mean temperature increases from west to east, then the thermal wind is from the south.