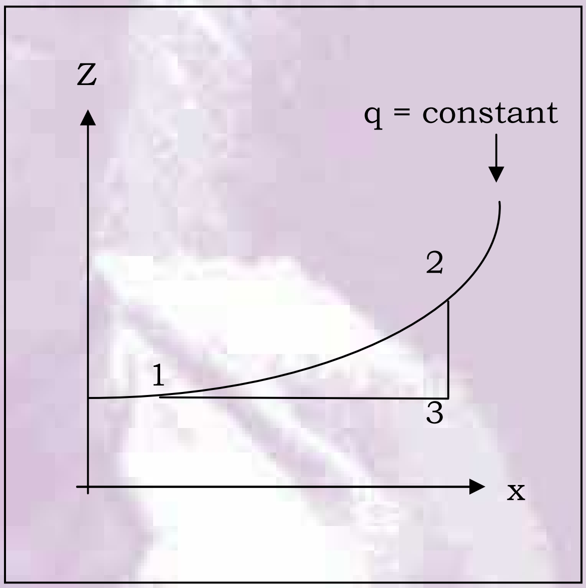
Pressure Gradient Force on Isobaric Surfaces

Up to this point, we have been using Cartesian co-ordinates. As a result, the vertical co-ordinate has been height, z. However, most meteorologists do not use height as a vertical co-ordinate because the governing equations are simpler when pressure, p, is the vertical co-ordinate. The reason for this is that the amount of mass between two isobaric surfaces is constant, and working with a constant mass allows fluid properties to be computed with greater accuracy. Furthermore, upper air weather maps are usually plotted on isobaric (or constant pressure) surfaces.

The equation of motion needs to be transformed from height to pressure. Luckily, only the pressure gradient force needs to be transformed – the rest of the equation does not change. The transformation equation is derived below for the arbitrary vertical co-ordinate, q, so that the results can be used for any vertical co-ordinate, not just pressure.



An east-west, vertical cross section through a surface of constant q is shown to the left. Points 1 and 2 are on the q surface; and point 3 is directly below point 2 and level with point 1. The pressure change from point 1 to point 2 is the same along any path: along the q surface from point 1 to point 2, or horizontally from point 1 to point 3 and then vertically to point 2.

$$\left(p_2 - p_1\right)_q = \left(p_3 - p_1\right)_z + \left(p_2 - p_3\right)_x$$

Divide both sides of this equation by $\Delta x = x_3 - x_1 = x_2 - x_1$

$$\left(\frac{p_2 - p_1}{\Delta x}\right)_q = \left(\frac{p_3 - p_1}{\Delta x}\right)_z + \left(\frac{p_2 - p_3}{\Delta x}\right)_x .$$

Multiply and divide the last term by, $\Delta z = z_2 - z_3$,

$$\left(\frac{p_2 - p_1}{\Delta x}\right)_q = \left(\frac{p_3 - p_1}{\Delta x}\right)_z + \left(\frac{p_2 - p_3}{\Delta z}\right)_x \frac{\Delta z}{\Delta x} .$$

As the finite differences become very small, in the limit,

$$\left(\frac{\partial p}{\partial x}\right)_q = \left(\frac{\partial p}{\partial x}\right)_z + \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_q .$$

Substituting from the hydrostatic equation,

$$\left(\frac{\partial p}{\partial x}\right)_q = \left(\frac{\partial p}{\partial x}\right)_z - \rho g\left(\frac{\partial z}{\partial x}\right)_q .$$

Divide both sides by the density,

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_q = \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z - g\left(\frac{\partial z}{\partial x}\right)_q .$$

Solve for the pressure gradient force on a level surface,

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_q - g\left(\frac{\partial z}{\partial x}\right)_q . \qquad (1)$$

Equation (1) can be used to transform the east-west component of the pressure gradient force on a level surface to any other vertical co-ordinate – just replace, q, with the desired vertical co-ordinate. For the north-south component, just replace each, x, with a, y. For example, to use pressure as the vertical co-ordinate, replace, q, with, p,

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_p - g\left(\frac{\partial z}{\partial x}\right)_p .$$

And since pressure does not change horizontally on an isobaric surface,

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -g\left(\frac{\partial z}{\partial x}\right)_p . \qquad (2)$$

Although there is no pressure gradient on an isobaric surface, the horizontal force known as the pressure gradient force does exist! Equation (2) shows that the horizontal pressure gradient force is directly proportional to the horizontal height gradient on an isobaric surface.

The same procedure can be used to find the pressure gradient force on other surfaces. For example, many text books consider an isentropic surface, or surface of constant potential temperature. The first step is to replace, q, with, θ, in equation (1),

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_\theta - g\left(\frac{\partial z}{\partial x}\right)_\theta . \qquad (3)$$

The last term on the RHS can easily be written as the gradient of gz. The first term on the RHS, however, will require some work to get into the needed form.

Recall the definition of potential temperature,

$$\theta \equiv T\left(\frac{1000}{p}\right)^{\frac{R_d}{c_p}} .$$

Take the natural log,

$$\ln\theta = \ln T + \left(\frac{R_d}{c_p}\right)\left(\ln 1000 - \ln p\right).$$

Take the partial derivative with respect to x, on a θ surface,

$$\left(\frac{\partial \ln\theta}{\partial x}\right)_\theta = \left(\frac{\partial \ln T}{\partial x}\right)_\theta - \left(\frac{R_d}{c_p}\right)\left(\frac{\partial \ln p}{\partial x}\right)_\theta .$$

Since the potential temperature is constant on a θ surface,

$$\left(\frac{\partial \ln T}{\partial x}\right)_\theta = \left(\frac{R_d}{c_p}\right)\left(\frac{\partial \ln p}{\partial x}\right)_\theta .$$

Expand the partial derivative of the natural logs,

$$\frac{1}{T}\left(\frac{\partial T}{\partial x}\right)_\theta = \left(\frac{R_d}{p\,c_p}\right)\left(\frac{\partial p}{\partial x}\right)_\theta .$$

Multiply by $T c_p$,

$$c_p\left(\frac{\partial T}{\partial x}\right)_\theta = \left(\frac{TR_d}{p}\right)\left(\frac{\partial p}{\partial x}\right)_\theta .$$

Use the equation of state,

$$c_p\left(\frac{\partial T}{\partial x}\right)_\theta = \left(\frac{1}{\rho}\right)\left(\frac{\partial p}{\partial x}\right)_\theta .$$

Substitute into equation (3),

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -c_p\left(\frac{\partial T}{\partial x}\right)_\theta - g\left(\frac{\partial z}{\partial x}\right)_\theta .$$

And finally,

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\frac{\partial}{\partial x}\left(c_p T + gz\right)_\theta .$$

Thus, the horizontal pressure gradient force is proportional to the horizontal gradient of dry static energy on an isentropic surface.