Linear Velocity Field

A simple way to study horizontal circulation (wind) is to ignore nonlinear effects, which are assumed to be smaller than linear effects. The result is called the linear velocity field.

Using a Cartesian coordinate system and the Taylor series to represent the wind components close to the origin,

$$u = u_o + \left(\frac{\partial u}{\partial x}\right)_o x + \left(\frac{\partial u}{\partial y}\right)_o y + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_o x^2 + \frac{1}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)_o y^2 + \left(\frac{\partial^2 u}{\partial x \partial y}\right)_o xy + \ldots$$

$$v = v_o + \left(\frac{\partial v}{\partial x}\right)_o x + \left(\frac{\partial v}{\partial y}\right)_o y + \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2}\right)_o x^2 + \frac{1}{2}\left(\frac{\partial^2 v}{\partial y^2}\right)_o y^2 + \left(\frac{\partial^2 v}{\partial x \partial y}\right)_o xy + \ldots$$

where the subscript zero indicates values at the origin, and x and y are distances from the origin. Near the origin x and y are small (<1), and products of x and y are even smaller (<<1). Thus, the non linear terms, with products of x and y, should be smaller than the linear terms and can be neglected,

$$u \approx u_o + \left(\frac{\partial u}{\partial x}\right)_o x + \left(\frac{\partial u}{\partial y}\right)_o y \qquad (1)$$

$$v \approx v_o + \left(\frac{\partial v}{\partial x}\right)_o x + \left(\frac{\partial v}{\partial y}\right)_o y \,. \qquad (2)$$

Equations (1) and (2) represent the linear velocity field, but a more useful set of equations can be obtained. The first step is to split the wind shear terms on the RHS into two parts,

$$u \approx u_o + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)_o x + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)_o x + \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)_o y + \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)_o y$$

$$v \approx v_o + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)_o x + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)_o x + \frac{1}{2}\left(\frac{\partial v}{\partial y}\right)_o y + \frac{1}{2}\left(\frac{\partial v}{\partial y}\right)_o y$$

Second, add zero to each equation by adding and subtracting the same quantities. For the u equation,

$$u \approx u_o + \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_o x + \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)_o x + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)_o y + \frac{1}{2}\left(-\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)_o y$$

For the v equation,

$$v \approx v_o + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)_o x + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_o x + \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_o y + \frac{1}{2}\left(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_o y$$

The quantities inside the parenthesis are significant properties of the horizontal wind and do not change when the axes are rotated, **rotationally invariant**,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \equiv \text{divergence,} \qquad \text{DIV,}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \equiv \text{vorticity,} \qquad \text{VOR,}$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \equiv \text{stretching deformation,} \qquad \text{DST,}$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \equiv \text{shearing deformation,} \qquad \text{DSH.}$$

And

$$u_o, v_o \equiv \text{translation.}$$

Finally,

$$u \approx u_o + \frac{1}{2} DIV_o\, x + \frac{1}{2} DST_o\, x + \frac{1}{2} DSH_o\, y - \frac{1}{2} VOR_o\, y$$

$$v \approx v_o + \frac{1}{2} DSH_o\, x + \frac{1}{2} VOR_o\, x + \frac{1}{2} DIV_o\, y - \frac{1}{2} DST_o\, y.$$

Thus, the linear velocity field is the sum of 5 components: translation, divergence, vorticity, stretching deformation, and shearing deformation.