Derivation of the Gradient Wind Equation

Although the geostrophic wind is a very useful concept, the derivation assumes that parcels move along straight lines. Since atmospheric flow is often curved, one must wonder if the geostrophic wind can be improved upon. To find out, we will start with the scaled equation of motion using height as the vertical coordinate,

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu.$$

Expand the total derivatives,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu.$$

Assume:

1. horizontal flow, $w = 0$, and
2. steady state, $\frac{\partial}{\partial t}(\ ) = 0$.

Then,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu.$$

Convert to polar coordinates:

1. horizontal advection terms,

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta};$$

2. wind components,

$$u = v_r\cos\theta - v_\theta\sin\theta$$

$$v = v_r \sin\theta + v_\theta \cos\theta;$$

3. spatial derivatives,

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}.$$

Then,

$$v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = f v_\theta - \frac{1}{\rho} \frac{\partial p}{\partial r}$$

$$v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} = - f v_r - \frac{1}{r\rho} \frac{\partial p}{\partial \theta}.$$

Assume symmetric (circular) pressure and wind fields,

$$\frac{\partial}{\partial \theta} (p, v_r, v_\theta) = 0.$$

Then,

$$v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} = f v_\theta - \frac{1}{\rho} \frac{\partial p}{\partial r}$$

$$v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_r}{r} = - f v_r.$$

Factoring a v_r from the last equation,

$$v_r \left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} + f \right) = 0.$$

The quantity within the parentheses is the sum of the relative vorticity and the Earth's vorticity, or the absolute vorticity. The Earth's vorticity is always positive, and the relative vorticity is positive in low pressure areas and negative in high pressure areas. The absolute vorticity is not zero; thus, v_r must be zero. This may seem unusual, but it is consistent with an assumption made earlier - that the wind is strictly horizontal. If there is no radial component to the wind, then there is no convergence into low pressure areas, and no vertical motion.

Finally, we have the **gradient wind equation**.

$$\frac{v_\theta^2}{r} + f\,v_\theta - \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$