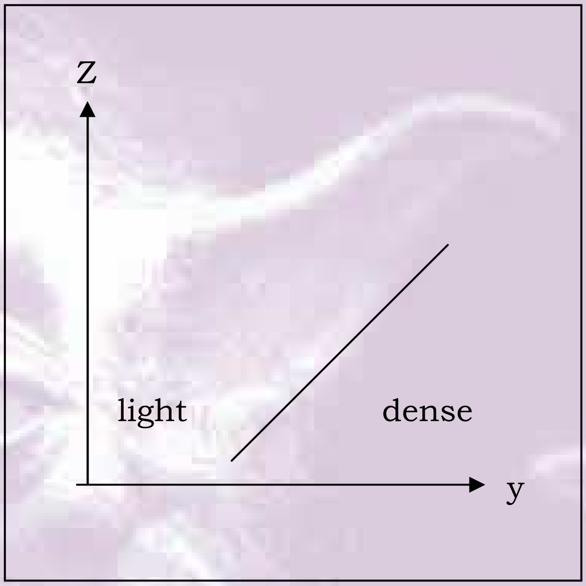
Slope of Fronts



An ideal front is often aligned east-west with dense air to the north, and light air to the south. The frontal boundary will have positive slope, dz/dy, since lighter air will be found above denser air, similar to the diagram at the left.

At the front, the **dynamic boundary condition**, $p_D - p_L = 0$, must be satisfied. This means that the pressure must be continuous across the boundary. If it were not continuous, then the pressure gradient and the pressure gradient force would be infinite, which would lead to an infinite wind. That is clearly impossible.

Since the pressure is a function of x, y, and z, the total differential of the pressure difference across the front can be obtained from the chain rule,

$$d(p_D - p_L) = \frac{\partial (p_D - p_L)}{\partial x} dx + \frac{\partial (p_D - p_L)}{\partial y} dy + \frac{\partial (p_D - p_L)}{\partial z} dz = 0,$$

or

$$d(p_D - p_L) = \left[\left(\frac{\partial p}{\partial x}\right)_D - \left(\frac{\partial p}{\partial x}\right)_L\right] dx + \left[\left(\frac{\partial p}{\partial y}\right)_D - \left(\frac{\partial p}{\partial y}\right)_L\right] dy + \left[\left(\frac{\partial p}{\partial z}\right)_D - \left(\frac{\partial p}{\partial z}\right)_L\right] dz = 0 .$$

Since the front is parallel to the x axis and the pressure on both sides must be the same for all values of x, the pressure gradient in the x direction, $\frac{\partial p}{\partial x}$, must be the same on both sides of the boundary, $\left(\frac{\partial p}{\partial x}\right)_D - \left(\frac{\partial p}{\partial x}\right)_L = 0$.

Then,

$$\left[\left(\frac{\partial p}{\partial y}\right)_D - \left(\frac{\partial p}{\partial y}\right)_L\right] dy + \left[\left(\frac{\partial p}{\partial z}\right)_D - \left(\frac{\partial p}{\partial z}\right)_L\right] dz = 0,$$

and,

$$-\frac{\left[\left(\frac{\partial p}{\partial y}\right)_D - \left(\frac{\partial p}{\partial y}\right)_L\right]}{\left[\left(\frac{\partial p}{\partial z}\right)_D - \left(\frac{\partial p}{\partial z}\right)_L\right]} = \frac{dz}{dy}. \tag{1}$$

Equation (1) gives the slope of the front in terms of the north-south and vertical pressure gradients on either side of the front. A more useful equation can be found by using the hydrostatic equation,

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.$$

Equation (1) becomes,

$$\frac{dz}{dy} = \frac{\left[\left(\frac{\partial p}{\partial y}\right)_D - \left(\frac{\partial p}{\partial y}\right)_L\right]}{g\left(\rho_D - \rho_L\right)}. \tag{2}$$

In equation (2), the denominator on the RHS is positive; and the LHS must be positive, because lighter air must lie above denser air. Thus, the numerator on the RHS must be positive as well, which means that the horizontal pressure gradient perpendicular to the front is larger on the dense side of the front than on the light side. As a result, isobars must "kink" at a front.

If the wind is assumed to be geostrophic,

$$\frac{\partial p}{\partial y} = -f_o u_g \rho,$$

equation (2) becomes,

$$\frac{dz}{dy} = \frac{f_o\left[\left(\rho u_g\right)_L - \left(\rho u_g\right)_D\right]}{g\left(\rho_D - \rho_L\right)}. \tag{3}$$

The numerator on the RHS is now expressed as a wind shear that represents relative vorticity. And since it must be positive, the relative vorticity along a front should be positive (cyclonic).

Next, substitute for the density from the equation of state,

$$\rho = \frac{p}{RT}.$$

Equation (3) becomes,

$$\frac{dz}{dy} = \frac{f_o\left[\left(p\,u_g/RT\right)_L - \left(p\,u_g/RT\right)_D\right]}{g\left(p/RT_D - p/RT_L\right)}.$$

Since the pressure is the same on both sides of the front,

$$\frac{dz}{dy} = \frac{f_o\left[\left(u_g/T\right)_L - \left(u_g/T\right)_D\right]}{g\left(1/T_D - 1/T_L\right)}.$$

Multiply the RHS by

$$\frac{T_L\ T_D}{T_L\ T_D},$$

$$\frac{dz}{dy} = \frac{f_o\left[T_D\left(u_g\right)_L - T_L\left(u_g\right)_D\right]}{g\left(T_L - T_D\right)}.$$

Since the percentage change in the temperature across a front is much less than the percentage change in the wind, a mean temperature can be used in the numerator,

$$\frac{dz}{dy} = \frac{f_o\,\overline{T}\left[\left(u_g\right)_L - \left(u_g\right)_D\right]}{g\left(T_L - T_D\right)}. \qquad (4)$$

Equation (4), which was first derived by Margules (1906), shows that the slope of a front is proportional to the cyclonic vorticity and inversely proportional to the temperature gradient perpendicular to the front.