Scalar Derivation of the Equation of Motion

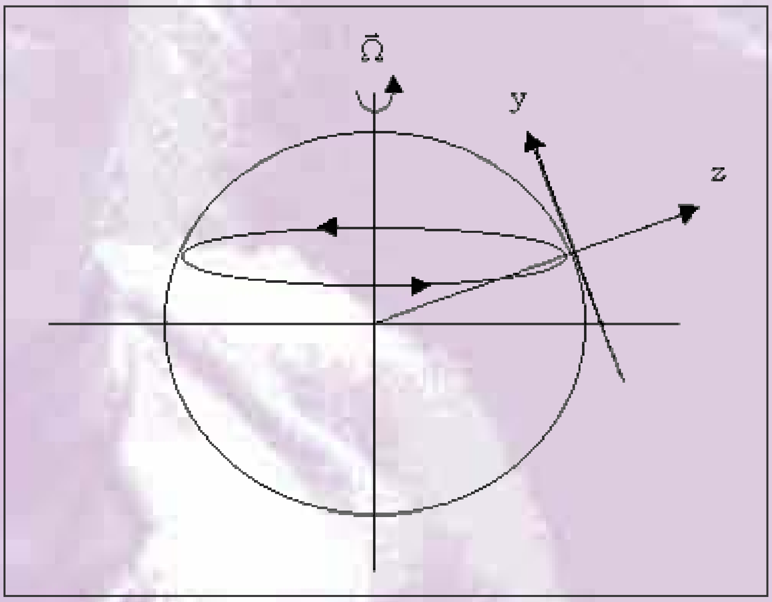
We often want to determine the acceleration of an air parcel that results from the sum of all forces acting upon it. The appropriate law of physics is **Newton's Second Law**,

$$F = m a . \qquad (1)$$

But in order to use Newton's Second Law in meteorology, an appropriate form, the **equation of motion**, must be derived.

Since most everything in meteorology is per unit mass, equation (1) is divided by the mass, m. Using the net force, F_{net}, which is just the sum of the forces,

$$a = \frac{F_{net}}{m} = \sum \frac{F}{m} .$$



The forces in Newton's Second Law are called **Newtonian forces**, and include the gravitational, pressure gradient, and frictional forces. Also, Newton's Second Law is only valid in an **inertial coordinate system** - a coordinate system that does not experience any accelerations.

An inertial coordinate system is rarely used in meteorology. Instead, we prefer to use a coordinate system that is attached to the surface of the Earth. As the Earth spins about its axis, the coordinate origin moves in a circle and each coordinate axis changes direction relative to the Sun. At noon the z axis points in the general direction of the Sun, while at midnight the z axis points in the general direction away from the Sun. Formally, this coordinate system is a **non-inertial coordinate system**.

The link between the two coordinate systems is that the acceleration in an inertial system, a_i, is equal to the acceleration in a non-inertial system, a_n, plus any apparent accelerations, a_a,

$$a_i = a_n + a_a.$$

The apparent accelerations explain the circular path of the non-inertial coordinate origin, which is a **centripetal acceleration**, a_{ctp} , and the rotation of the non-inertial coordinate axes, which is a **spin acceleration** , a_{spn} ,

$$a_i = a_n + a_{ctp} + a_{spn}.$$

Solving for the non-inertial acceleration,

$$a_n = a_i - a_{ctp} - a_{spn} .$$

The opposite of the centripetal acceleration is the **centrifugal acceleration**, a_{cfg} , and the opposite of the spin acceleration is the **Coriolis acceleration**, a_{cor} ,

$$a_n = a_i + a_{cfg} + a_{cor} .$$

Introducing the accelerations due to the Newtonian forces: pressure gradient force, PGF, gravity, G, and friction, F,

$$a_n = PGF + G + F + a_{cfg} + a_{cor} .$$

The sum of Newtonian gravity and the centrifugal acceleration yield **apparent gravity**, g,

$$a_n = PGF + g + F + a_{cor} . \qquad (2)$$

In meteorology, the most often used coordinate system is a **Cartesian coordinate system**, with the x axis pointing east and the y axis pointing north. The z axis is chosen in the opposite direction of apparent gravity, which points up. (The angle between apparent and Newtonian gravity is less than 0.1^{o}, and apparent gravity is more nearly perpendicular to the surface of an oblate spheroid such as the Earth.)

Equation (2) may be written as an equation along each axis,

$$\text{x:} \quad \frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x + 2\Omega\left(v\sin\phi - w\cos\phi\right)$$

$$\text{y:} \quad \frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y - 2\Omega\, u\sin\phi$$

$$\text{z:} \quad \frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z + 2\Omega\, u\cos\phi .$$

When these equations are applied to synoptic-scale circulations, several terms are significantly smaller than the other terms, and are neglected. The resulting **scaled equation of motion** is:

x: $$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega\, v \sin\phi$$

y: $$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega\, u \sin\phi$$

z: $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.$$

Introducing the **Coriolis parameter**, $f = 2\Omega \sin\phi$,

x: $$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

y: $$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu$$

z: $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.$$