Thermodynamic Energy Equation

Start with the First Law of Thermodynamics,

$$dQ = c_p\, dT - \alpha\, dp.$$

Divide by dt ,

$$\frac{dQ}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt}.$$

Expand the total derivative of temperature using pressure as the vertical co-ordinate,

$$\frac{dQ}{dt} = c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + \omega \frac{\partial T}{\partial p} \right) - \alpha \frac{dp}{dt}.$$

Since,

$$\frac{dp}{dt} \equiv \omega,$$

$$\frac{dQ}{dt} = c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) + \omega \left(c_p \frac{\partial T}{\partial p} - \alpha \right).$$

Multiply both sides by $-\dfrac{1}{c_p}$,

$$-\frac{1}{c_p} \frac{dQ}{dt} = -\left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) + \omega \left(\frac{\alpha}{c_p} - \frac{\partial T}{\partial p} \right).$$

Solve for the local change of temperature with time,

$$\frac{\partial T}{\partial t} = -\left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) + \omega \left(\frac{\alpha}{c_p} - \frac{\partial T}{\partial p} \right) + \frac{1}{c_p} \frac{dQ}{dt}. \qquad (1)$$

Equation (1) shows that the local temperature change is a result of horizontal temperature advection,

$$-\left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right),$$

adiabatic compression,

$$\omega \frac{\alpha}{c_p},$$

vertical temperature advection,

$$-\omega \frac{\partial T}{\partial p},$$

and diabatic heating,

$$\frac{1}{c_p} \frac{dQ}{dt}.$$