Total and Partial Derivatives

The total change of any dependent variable (meteorological quantity) is the sum of the changes with respect to each independent variable (time and spatial quantities), according to the chain rule. If the temperature, T, is a function of x, y, z, and t, the total change of temperature, dT, is

$$dT = \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy + \frac{\partial T}{\partial z}\,dz + \frac{\partial T}{\partial t}\,dt.$$

Divide by the change in time, dt,

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} + \frac{\partial T}{\partial t}\frac{dt}{dt}.$$

Since,

$$\frac{dx}{dt} = u,\ \frac{dy}{dt} = v, \text{ and } \frac{dz}{dt} = w,$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\,u + \frac{\partial T}{\partial y}\,v + \frac{\partial T}{\partial z}\,w + \frac{\partial T}{\partial t}\,.$$

To make it clear that the wind components are multiplied by the partial derivatives and not to be differentiated, write the wind components first,

$$\boxed{\frac{dT}{dt} = u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} + w\,\frac{\partial T}{\partial z} + \frac{\partial T}{\partial t}}\ . \qquad (1)$$

While equation (1) was derived for the temperature, it can be used to expand the total derivative of any dependent variable. Everywhere a, T, appears in equation (1), replace it with whatever dependent variable you are using.

If the dependent variable is not a function of height, z, but pressure, p, then use the differential operator $\omega\,\frac{\partial}{\partial p}$ instead of $w\,\frac{\partial}{\partial z}$.

The symbol, d, usually appears in the **total derivative**, while the symbol, ∂, usually appears in the **partial derivative**. Clearly, the total and partial derivatives with respect to time are different; one is equal to the other plus three additional terms. Those three terms are called the **advective terms**, because they are not exactly equal to the advection.

The partial derivative with respect to time is called the **tendency**, and represents the change of a dependent variable with time *at a fixed location*. An example is the common 3 hour surface pressure tendency at a weather station.

The total derivative with respect to time is called the total change, and represents the change of a dependent variable with time *following a moving mass,* such as an air parcel. An example is wind velocity, which is the change in position with time of a moving parcel.

To see which processes can change the surface temperature at a weather station, solve equation (1) for the temperature tendency,

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} - w\frac{\partial T}{\partial z} + \frac{dT}{dt}.$$

The first two terms on the right hand side, RHS, are the **horizontal temperature advection**. The third term on the RHS is the **vertical temperature advection**. The sum of these three terms are the **total temperature advection**.

The fourth term on the RHS is the temperature change due to all other processes that affect the moving parcel; two examples are radiation and adiabatic cooling due to expansion. To verify this, divide the First Law of Thermodynamics by dt,

$$\frac{dQ}{dt} = c_p\frac{dT}{dt} + g\frac{dz}{dt}.$$

Solve for the total change of temperature with time,

$$\frac{dT}{dt} = \frac{1}{c_p}\frac{dQ}{dt} - \frac{g}{c_p}\frac{dz}{dt}.$$

Then,

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} - w\frac{\partial T}{\partial z} + \frac{1}{c_p}\frac{dQ}{dt} - \frac{g}{c_p}\frac{dz}{dt},$$

or,

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} - w\frac{\partial T}{\partial z} - w\frac{g}{c_p} + \frac{1}{c_p}\frac{dQ}{dt}.$$

Thus, vertical motion contributes to temperature change through two processes: term three – vertical advection, and term four – adiabatic compression. Term four is usually larger (in absolute value) than term three, since the lapse rate is usually less than the dry adiabatic process rate. And, subsidence will tend to produce warming.